BOOST-INVARIANT PERFECT FERMI-DIRAC SPIN HYDRODYNAMICS

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December 14, 2025

MOTIVATION

1. Non-zero spin polarization of Lambda hyperons and vector mesons produced in high-energy heavy-ion collisions $\longrightarrow$ **spin hydrodynamics**.

2. Non-local collisions $\longrightarrow$ correspondence between orbital and spin parts of total angular momentum.

3. As a system obtains global equilibrium $\longrightarrow$ **spin polarization tensor $\omega_{\mu\nu}$**.

4. **Hybrid approach** - description of PF spin hydrodynamics with classical spin and with introduction of dissipative effects through the entropy-based Israel-Stewart method
   $\longrightarrow$ sidestepping the complexities of the non-local collision formalism.

5. In natural units $\omega_{\mu\nu} = \frac{\Omega_{\mu\nu}}{T}$ $\longrightarrow$ expansion in components of $\omega_{\mu\nu}$ is well defined.

6. When <u>at least</u> second-order terms in $\omega_{\mu\nu}$ are included (specifically into N^{μ} and $T^{\mu\nu}$), consistency across different spin hydrodynamics formulations is obtained.

7. Simplest system: one-dimensional, boost invariant expansion without dissipative corrections (extension of the articles from 2019 [1] and then 2025 [2]).

[1] W. Florkowski, A. Kumar, R. Ryblewski, and R. Singh, Phys.Rev.C 99 (2019).
[2] Z. Drogosz, W. Florkowski, NŁ, R. Ryblewski, Phys.Rev.C 111 (2025).

Framework for boost-invariant and transversely homogeneous system

- Orthonormal basis

$$\begin{aligned} U^\alpha &= \frac{1}{\tau}(t,0,0,z) = (\cosh\eta, 0, 0, \sinh\eta)\,, \\ X^\alpha &= (0,1,0,0), & \tau &= \sqrt{t^2 - z^2}, \\ Y^\alpha &= (0,0,1,0), & \eta &= \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right), \\ Z^\alpha &= \frac{1}{\tau}(z,0,0,t) = (\sinh\eta, 0, 0, \cosh\eta)\,. \end{aligned} \tag{1}$$

- Transformation rule for derivatives

$$\begin{bmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} = \begin{bmatrix} \cosh\eta & 0 & 0 & -\frac{1}{\tau}\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \frac{1}{\tau}\cosh\eta \end{bmatrix} \begin{bmatrix} \partial_\tau \\ \partial_x \\ \partial_y \\ \partial_\eta \end{bmatrix}. \tag{2}$$

- Projection operator

$$\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu = -X^\mu X^\nu - Y^\mu Y^\nu - Z^\mu Z^\nu. \tag{3}$$

SPIN POLARIZATION TENSOR $\omega_{\mu\nu}$

- Ranked-2 antisymmetric tensor, decomposed to electriclike and magneticlike four-vectors

$$\omega_{\mu\nu} = k_\mu U_\nu - k_\nu U_\mu + \epsilon_{\mu\nu\alpha\beta} U^\alpha \omega^\beta, \tag{4}$$

$$t_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} U^\alpha \omega^\beta, \qquad t^\mu = t^{\mu\nu} k_\nu = \epsilon^{\mu\nu\alpha\beta} k_\nu U_\alpha \omega_\beta. \tag{5}$$

- Orthogonality relations

$$k_\mu U^\mu = 0, \qquad \omega_\mu U^\mu = 0, \qquad t_\mu U^\mu = 0. \tag{6}$$

- Decomposition in the orthonormal basis

$$\begin{aligned} k^\mu &= C_{kx} X^\mu + C_{ky} Y^\mu + C_{kz} Z^\mu = \left(C_{kz} \sinh\eta, C_{kx}, C_{ky}, C_{kz} \cosh\eta\right), \\ \omega^\mu &= C_{\omega x} X^\mu + C_{\omega y} Y^\mu + C_{\omega z} Z^\mu = \left(C_{\omega z} \sinh\eta, C_{\omega x}, C_{\omega y}, C_{\omega z} \cosh\eta\right), \\ t^\mu &= V_x X^\mu + V_y Y^\mu + V_z Z^\mu = \left(V_z \sinh\eta, V_x, V_y, V_z \cosh\eta\right), \end{aligned} \tag{7}$$

where the coefficients $C_{ki} = C_{ki}(\tau),\ C_{\omega i} = C_{\omega i}(\tau)$, $i \in \{x, y, z\}$ and $\boldsymbol{V} \equiv \boldsymbol{C}_k \times \boldsymbol{C}_\omega = (V_x, V_y, V_z)$.

1. **Baryon number conservation** $\rightarrow$ $\partial_\mu N^\mu(x) = 0$,
   with the tensor form [3] [4]

$$N^\mu = \underbrace{\left(n_0 + n_2^\omega + n_2^k\right)}_{n_u} u^\mu + n_t t^\mu . \tag{8}$$

$$\Downarrow$$

$$\frac{\mathrm{d}n_u}{\mathrm{d}\tau} + \frac{n_u}{\tau} = 0. \tag{9}$$

Eq. (9) has a scaling solution

$$n_u\left(T(\tau), \xi(\tau), k^2(\tau), \omega^2(\tau)\right) = \frac{\tau_0}{\tau}\, n_u\left(T(\tau_0), \xi(\tau_0), k^2(\tau_0), \omega^2(\tau_0)\right), \tag{10}$$

where $\xi = \frac{\mu}{T}$ and the coefficients n_0, n_t, n_2^ω, n_2^{k} are functions of T, ξ, k^2, ω^2.

[3] W. Florkowski, M. Hontarenko, Phys.Rev.Lett. 134 (2025).
[4] Z. Drogosz, W. Florkowski, M. Hontarenko, Phys.Rev.D 110 (2024).

2. **Energy and linear momentum conservation** $\rightarrow$ $\boldsymbol{\partial_\mu T^{\mu\nu} = 0}$,
   with energy-momentum tensor

$$T^{\mu\nu} = \underbrace{\left(\varepsilon_0 + \varepsilon_2^{\omega} + \varepsilon_2^{k}\right)}_{\varepsilon_u} u^\mu u^\nu - \underbrace{\left(P_0 + P_2^{\omega} + P_2^{k}\right)}_{P_\Delta} \Delta^{\mu\nu} + P_t \left(t^\mu u^\nu + t^\nu u^\mu\right) + P_{k\omega}\left(k^\mu k^\nu + \omega^\mu \omega^\nu\right), \tag{11}$$

$$\Downarrow$$

$$\begin{aligned} \partial_\mu T^{\mu\nu} = &\left[\dot{\varepsilon}_u + \frac{\varepsilon_u + P_\Delta}{\tau} + \frac{P_{k\omega}}{\tau}\left(C_{kz}^2 + C_{\omega z}^2\right)\right] U^\nu + \left[\left(\dot{P}_t + \frac{P_t}{\tau}\right) V_x + P_t \dot{V}_x\right] X^\nu \\ &+ \left[\left(\dot{P}_t + \frac{P_t}{\tau}\right) V_y + P_t \dot{V}_y\right] Y^\nu + \left[\left(\dot{P}_t + \frac{P_t}{\tau}\right) V_z + P_t\left(\dot{V}_z + \frac{V_z}{\tau}\right)\right] Z^\nu = 0, \end{aligned} \tag{12}$$

where the coefficients ε_u, P_Δ, P_t and $P_{k\omega}$ are functions of T, ξ, k^2 and ω^2.

CONSERVATION LAWS FOR PF HYDRODYNAMICS - **SPIN PART OF THE ANGULAR MOMENTUM**

3. **Conservation of the spin part of the angular momentum** $\rightarrow$ $\boldsymbol{\partial_\lambda S^{\lambda,\mu\nu} = 0}$,
   with spin tensor

$$S^{\lambda,\mu\nu} = U^\lambda \left[A_3 \left(k^\mu U^\nu - k^\nu U^\mu\right) + A_1 t^{\mu\nu}\right] + \frac{A_3}{2}\left[\left(t^{\lambda\mu} U^\nu - t^{\lambda\nu} U^\mu\right) + \left(\Delta^{\lambda\mu} k^\nu - \Delta^{\lambda\nu} k^\mu\right)\right], \quad (13)$$

$$\Downarrow$$

$$\begin{aligned} \partial_\lambda S^{\lambda,\mu\nu} = & \frac{1}{\tau}\left[A_3 \left(k^\mu U^\nu - k^\nu U^\mu\right) + A_1 t^{\mu\nu}\right] + \dot{(A_3 k^\mu)} U^\nu - \dot{(A_3 k^\nu)} U^\mu \\ & + \dot{A}_1 t^{\mu\nu} + A_1 \dot{t}^{\mu\nu} + \frac{1}{2}\partial_\lambda A_3 \left(t^{\lambda\mu} U^\nu - t^{\lambda\nu} U^\mu\right) \\ & + \frac{A_3}{2}\left[\left(\partial_\lambda t^{\lambda\mu}\right) U^\nu + t^{\lambda\mu}\partial_\lambda U^\nu - \left(\partial_\lambda t^{\lambda\nu}\right) U^\mu - t^{\lambda\nu}\partial_\lambda U^\mu \right. \\ & \left. - \frac{U^\mu}{\tau} k^\nu + \Delta^{\lambda\mu}\partial_\lambda k^\nu + \frac{U^\nu}{\tau} k^\mu - \Delta^{\lambda\nu}\partial_\lambda k^\mu\right] = 0, \end{aligned} \quad (14)$$

where A_1 and A_3 are scalar functions depending only on τ.

CONSERVATION LAWS FOR PF HYDRODYNAMICS - **SPIN PART OF THE ANGULAR MOMENTUM**

Solution to (14) is obtained by contracting it with tensors $X_\mu Y_\nu$, $Y_\mu Z_\nu$, $Z_\mu X_\nu$, $U_\mu X_\nu$, $U_\mu Y_\nu$, and $U_\mu Z_\nu$ separately and have the form

$$\begin{bmatrix} A_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_1 \end{bmatrix} \begin{bmatrix} \dot{C}_{kx} \\ \dot{C}_{ky} \\ \dot{C}_{kz} \\ \dot{C}_{\omega x} \\ \dot{C}_{\omega y} \\ \dot{C}_{\omega z} \end{bmatrix} = \begin{bmatrix} Q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_2 \end{bmatrix} \begin{bmatrix} C_{kx} \\ C_{ky} \\ C_{kz} \\ C_{\omega x} \\ C_{\omega y} \\ C_{\omega z} \end{bmatrix}, \tag{15}$$

where

$$\begin{aligned} Q_1 &= -\left[\dot{A}_3 + \frac{1}{\tau}\left(A_3 + \frac{1}{2}A_3\right)\right], \qquad & Q_2 &= -\left(\dot{A}_3 + \frac{A_3}{\tau}\right), \\ R_1 &= -\left[\dot{A}_1 + \frac{1}{\tau}\left(A_1 - \frac{1}{2}A_3\right)\right], \qquad & R_2 &= -\left(\dot{A}_1 + \frac{A_1}{\tau}\right). \end{aligned} \tag{16}$$

- **Result**: 11 eqs. for 8 unknown functions $\Rightarrow$ overdetermined system of equations.
- **Constraint**: additional symmetry $\Rightarrow$ $\boldsymbol{V} = \boldsymbol{C}_k \times \boldsymbol{C}_\omega = 0$.

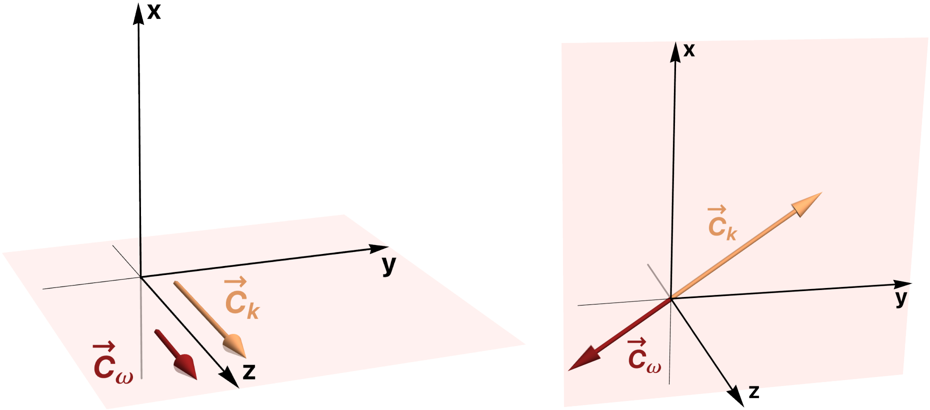


FIGURE: Schematic view of the longitudinal (left panel) and the transverse (right panel) configuration.

1. **Longitudinal configuration:** $\boldsymbol{C}_k = (0,0,C_{kz}), \quad \boldsymbol{C}_\omega = (0,0,C_{\omega z}),$

$$\Downarrow$$

$$\begin{aligned} \dot{n}_u &= -\frac{n_u}{\tau}, \\ \dot{\varepsilon}_u &= -\frac{\varepsilon_u + P_\Delta}{\tau} - \frac{P_{k\omega}}{\tau}\left(C_{kz}^2 + C_{\omega z}^2\right), \\ \dot{C}_{kz} &= -\left(\frac{\dot{A}_3}{A_3} + \frac{1}{\tau}\right) C_{kz}, \\ \dot{C}_{\omega z} &= -\left(\frac{\dot{A}_1}{A_1} + \frac{1}{\tau}\right) C_{\omega z}, \end{aligned} \tag{17}$$

$$\Downarrow$$

$$C_{kz}(\tau) = \frac{C_{kz}^0 A_3^0 \tau_0}{A_3\left(T(\tau), \xi(\tau)\right)\tau}, \qquad C_{\omega z}(\tau) = \frac{C_{\omega z}^0 A_1^0 \tau_0}{A_1\left(T(\tau), \xi(\tau)\right)\tau}. \tag{18}$$

2. **Transverse configuration:** $\boldsymbol{C}_k = (C_{kx}, C_{ky}, 0)\,, \quad \boldsymbol{C}_\omega = \lambda \boldsymbol{C}_k,$

$$\Downarrow$$

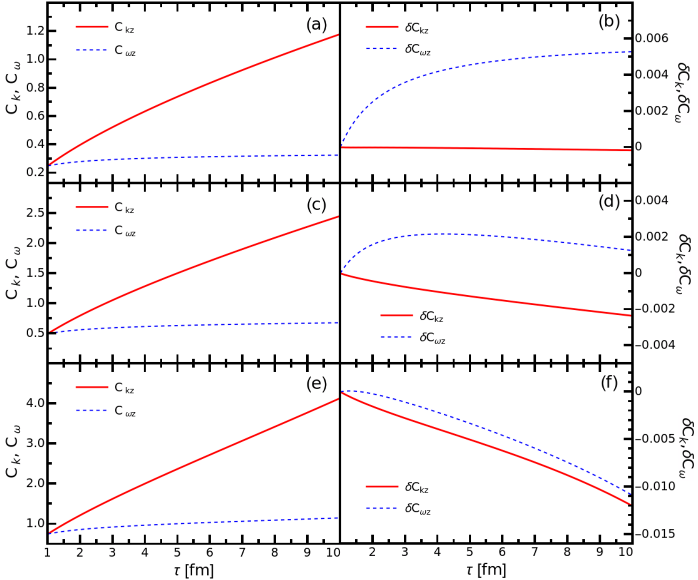
$$\begin{aligned} \dot{n}_u &= -\frac{n_u}{\tau}, \qquad \dot{\varepsilon}_u = -\frac{\varepsilon_u + P_\Delta}{\tau}, \\ \dot{C}_{kx} &= -\left(\frac{\dot{A}_3}{A_3} + \frac{3}{2\tau}\right) C_{kx}, \qquad \dot{C}_{ky} = -\left(\frac{\dot{A}_3}{A_3} + \frac{3}{2\tau}\right) C_{ky}, \\ \dot{C}_{\omega x} &= -\left(\frac{\dot{A}_1}{A_1} + \frac{1}{\tau}\left(1 - \frac{A_3}{2A_1}\right)\right) C_{\omega x}, \qquad \dot{C}_{\omega y} = -\left(\frac{\dot{A}_1}{A_1} + \frac{1}{\tau}\left(1 - \frac{A_3}{2A_1}\right)\right) C_{\omega y}, \end{aligned} \tag{19}$$

$$\Downarrow$$

$$C_{kx} = \frac{C^0_{kx} A^0 \tau_0}{A(T(\tau), \xi(\tau))\tau^{3/2}}, \qquad C_{ky} = \frac{C^0_{ky} A^0 \tau_0}{A(T(\tau), \xi(\tau))\tau^{3/2}}. \tag{20}$$

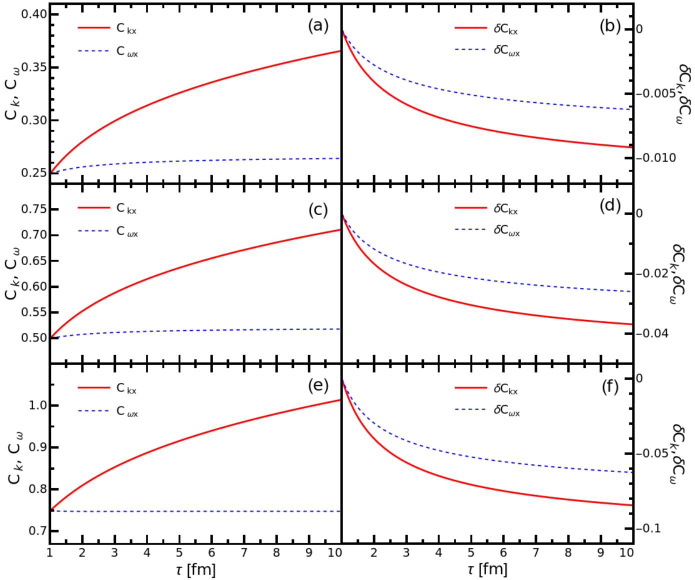
Note: $\lambda = \lambda(\tau)$ and $\lambda(\tau_0) = 1$.

$m = 1.116$ GeV, $T_0 = 155$ MeV, $\mu_0 = 800$ MeV, $\tau_0 = 1$ fm, $\tau_f = 10$ fm, $\quad \delta C_{iz} = \frac{C_{iz,FD}}{C_{iz,B}} - 1$ for $i \in \{k, \omega\}$



NUMERICAL RESULTS - **TRANSVERSE CONFIGURATION**

$$\delta C_{ix} = \frac{C_{ix,FD}}{C_{ix,B}} - 1 \quad \text{for} \quad i \in \{k, \omega\}$$



The perfect spin hydrodynamics applicability criterion [5]

$$\mathfrak{s}\sqrt{\boldsymbol{b}'^2 + \boldsymbol{e}'^2 + 2|\boldsymbol{e}' \times \boldsymbol{b}'|} < \frac{m}{T}, \tag{21}$$

where primes denotes quantities in LRF of the fluid element, $\mathfrak{s} = \sqrt{3/4}$ is the normalization of the spin four-vector, $\boldsymbol{e} = (e^1, e^2, e^3)$ and $\boldsymbol{b} = (b^1, b^2, b^3)$ are electriclike and magneticlike three-vectors

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}. \tag{22}$$

For **boost-invariant** geometry, eq.(21) takes the form

$$\mathfrak{s}\sqrt{\boldsymbol{C}_k^2(\tau) + \boldsymbol{C}_\omega^2(\tau) + 2\,|\boldsymbol{C}_k(\tau) \times \boldsymbol{C}_\omega(\tau)|} < \frac{m}{T(\tau)}. \tag{23}$$

[5] Z. Drogosz, W. Florkowski, and V. Mykhaylova, Phys.Rev.D 112 (2025).

Applicability condition: convergence of the integral appearing in the generating function

$$n_{\rm cl} = 2 \int \mathrm{d}P \cosh\xi \exp\left(-p\cdot\beta\right) \int \mathrm{d}S \underbrace{\exp\left(\frac{1}{2}\omega : s\right)}_{1+\frac{\omega:s}{2}+\frac{(\omega:s)^2}{8}} . \tag{24}$$

- **Full theory:** integral in (24) converges.
- **Expansion to the second-order in $\omega_{\mu\nu}$:** the second integral (over $\mathrm{d}S$) may become negative.

$$\Downarrow$$

$$\sqrt{\boldsymbol{C}_k^2(\tau) + \boldsymbol{C}_\omega^2(\tau) + 2\left|\boldsymbol{C}_k(\tau) \times \boldsymbol{C}_\omega(\tau)\right|} < 4, \tag{25}$$

for typical values of T and m, this application range is narrower than that of the full theory.

- **Expansion to the fourth-order in $\omega_{\mu\nu}$:** there is no problem with convergence (integral over $\mathrm{d}S$ is positive).

$$\Downarrow$$

It seems interesting to look at the behavior of the system expanded to the 4th order expansion.

CONCLUSIONS

1. Second-order corrections in spin polarization tensor $\longrightarrow$ feedback of the spin degrees of freedom.
2. We considered such corrections in one-dimensional and boost-invariant expansion, for particles with spin-$1/2$, described by Fermi–Dirac statistic rather than Boltzmann approximation.
3. Calculations resulted with overdetermined system $\longrightarrow$ additional symmetry constraints.
4. We constructed mathematically allowed "longitudinal" and "transverse" configurations.
5. The Fermi-Dirac corrections led to relative differences in spin polarization of up to about 8.5% and are bigger for the higher initial values of the spin polarization tensor coefficients.
6. Using the exact Fermi-Dirac statistics instead of its Boltzmann approximation did not prevent the breakdown of the considered model for high initial magnitudes of $\omega_{\mu\nu}$.
7. Future aspects:
    - looking at the behavior of system expanded up to the higher order or even exact expressions for $\omega_{\mu\nu}$,
    - including dissipative corrections witch could led to the stable behavior of the system.