

Exact Wigner function for chiral spirals

by Sudip Kumar Kar¹

Collaborators: Samapan Bhadury ,Wojciech Florkowski
and Valeriya Mykhaylova

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At

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Strange and Heavy Flavour Physics

14th December 2025



Motivation

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Physics Letters B

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Letter

Polarization of spin- $1/2$ particles with effective spacetime dependent masses

Samapan Bhadury^a✉, Arpan Das^{b c}✉, Wojciech Florkowski^a✉, Gowthama K.K.^d✉, Radoslaw Ryblewski^b✉

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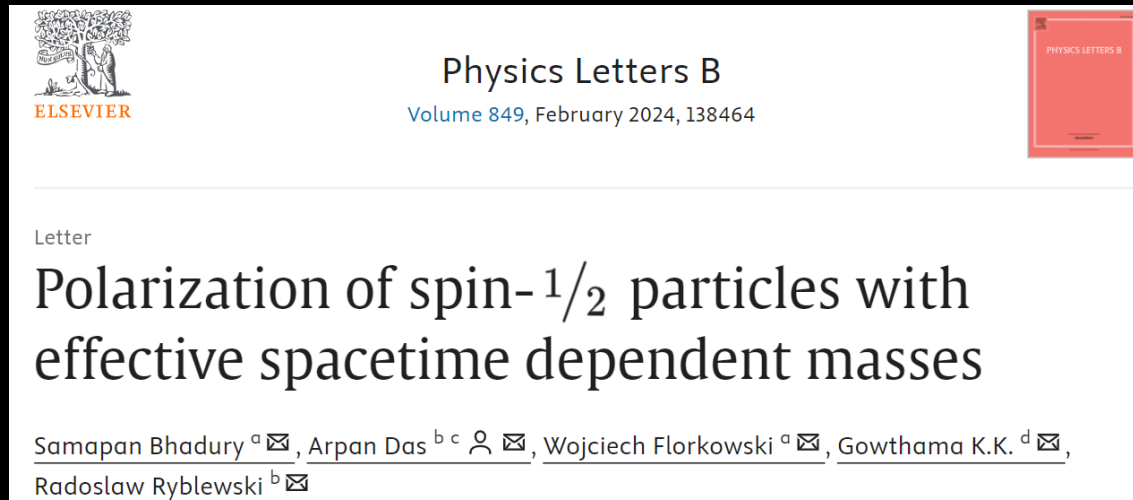
Samapan Bhadury ^a✉, Arpan Das ^{b c} ✉, Wojciech Florkowski ^a✉, Gowthama K.K. ^d✉, Radoslaw Ryblewski ^b✉



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Spin density *Mass(time dependent)*

Motivation

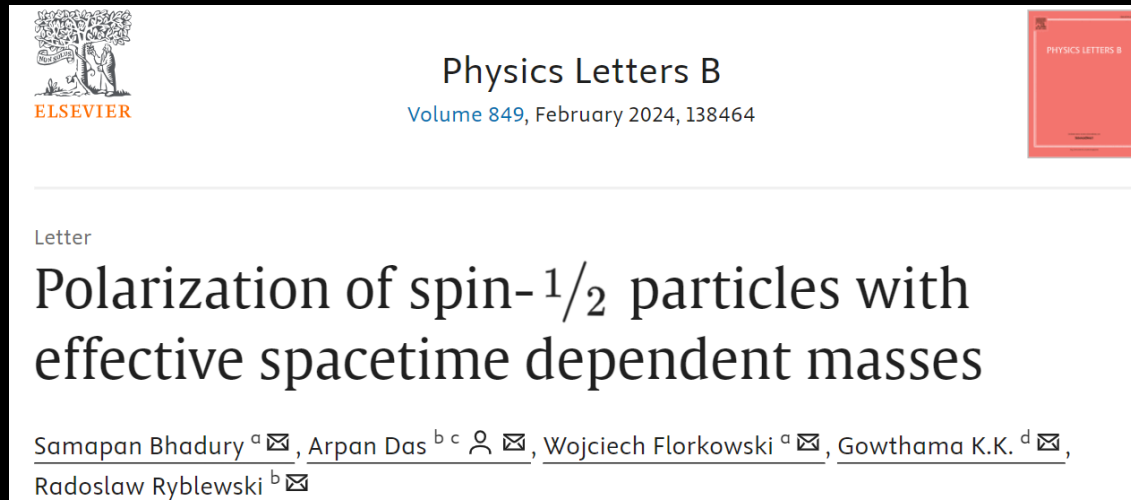


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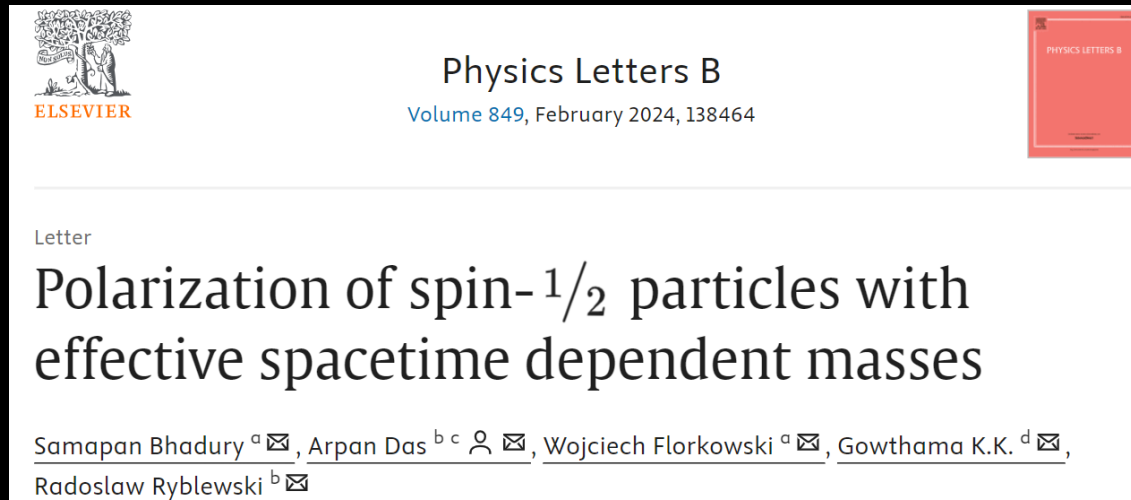
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- Thus, chiral spiral was selected

Chiral spirals as a potential ground state of matter

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Pion condensation in a relativistic field theory consistent with bulk properties of nuclear matter ☆

B. Banerjee[†], N.K. Glendenning, M. Gyulassy

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**QUARKS WITH A PION CONDENSATE
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B.

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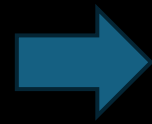
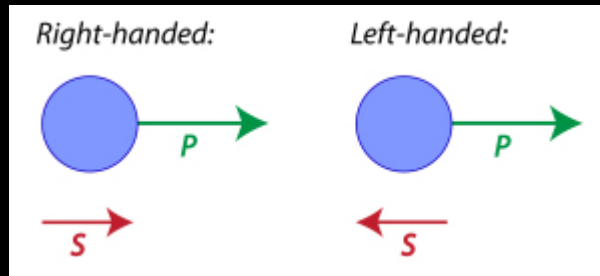
B. Marek Kut

How neutron star properties disfavor a nuclear chiral density wave

Orestis Papadopoulos* and Andreas Schmitt 

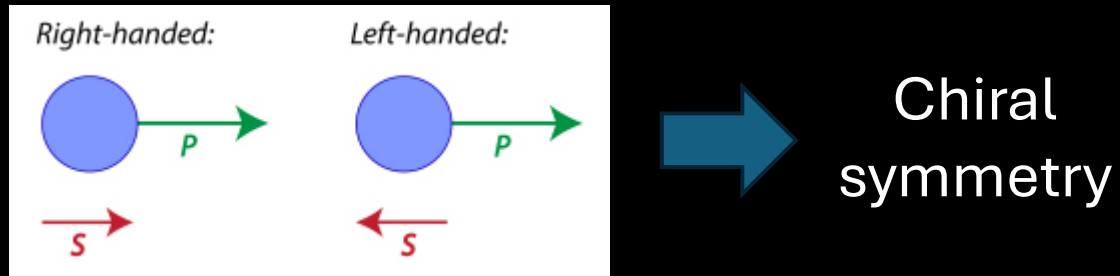
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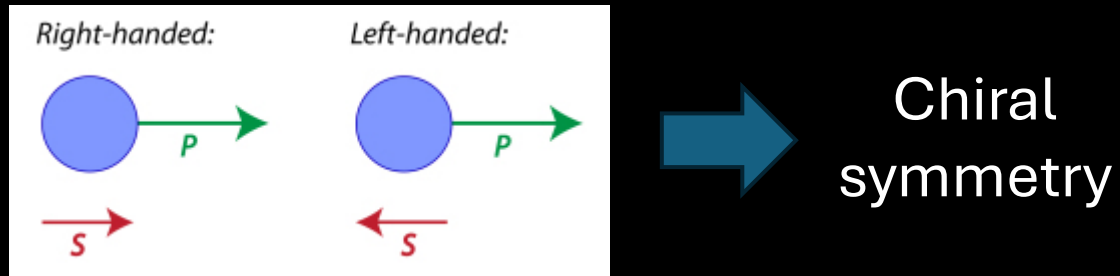
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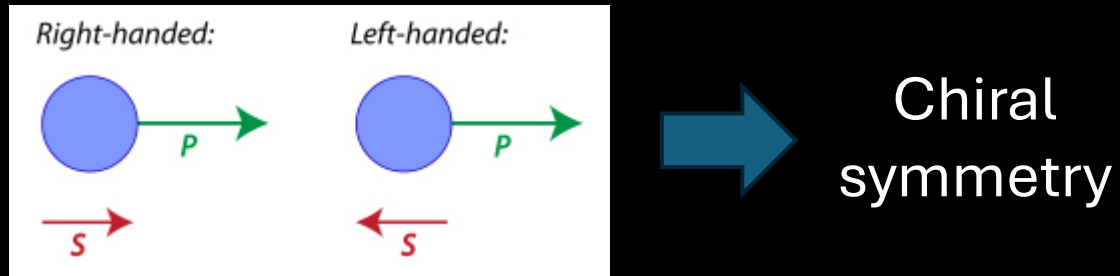
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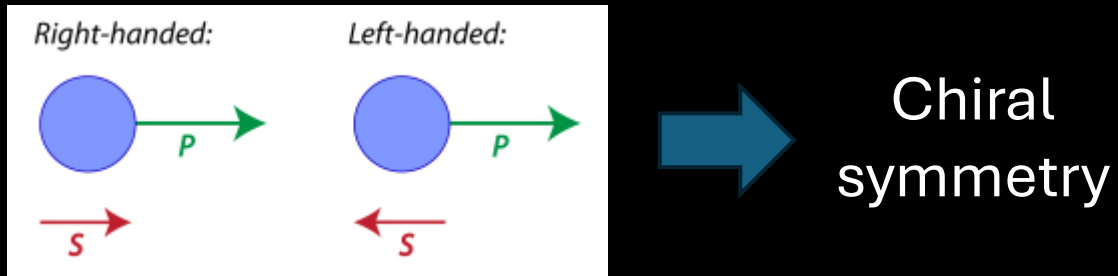
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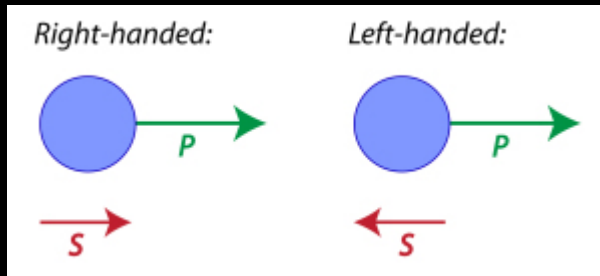
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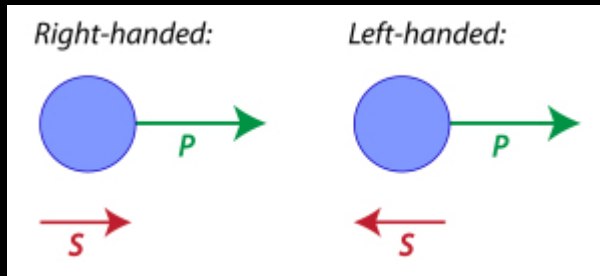


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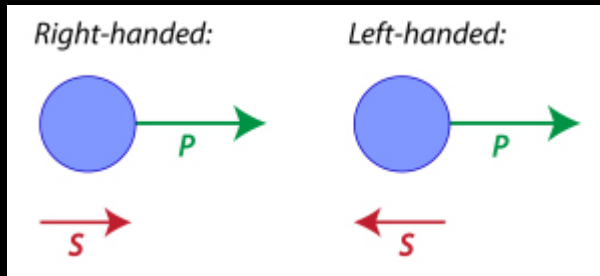
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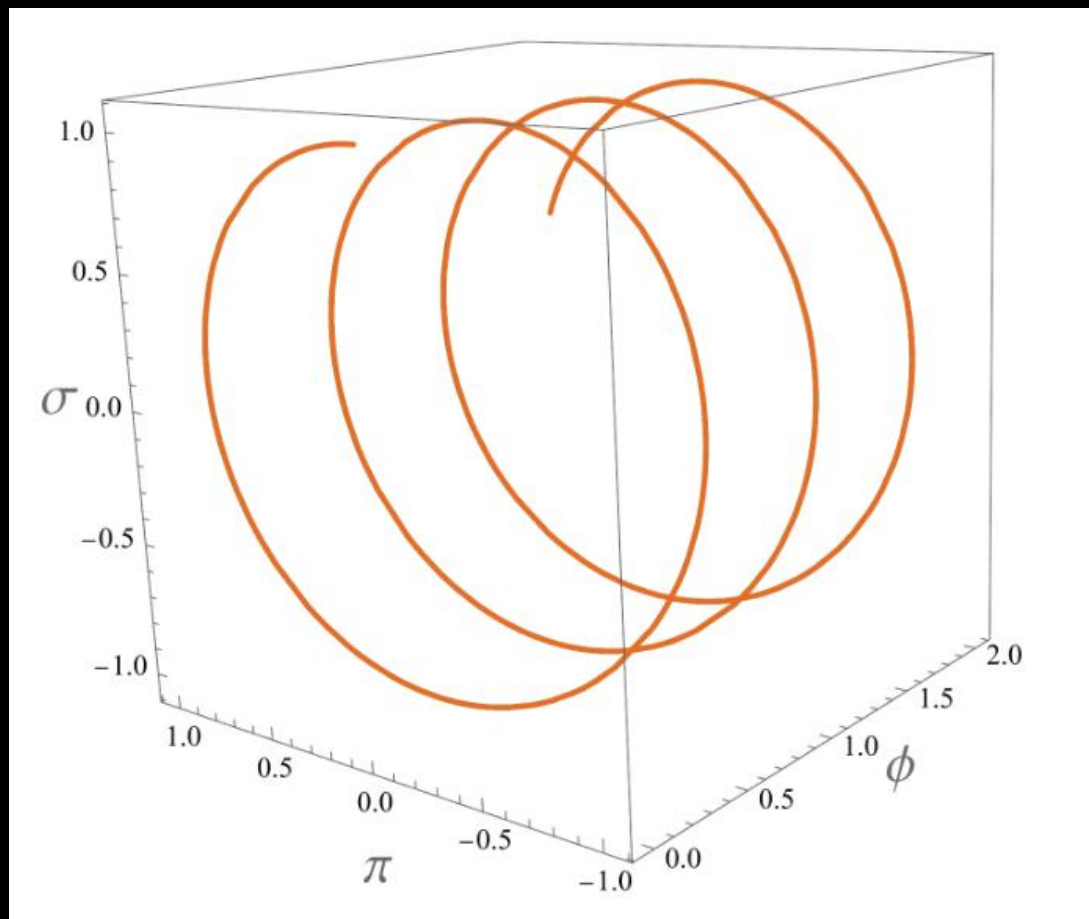
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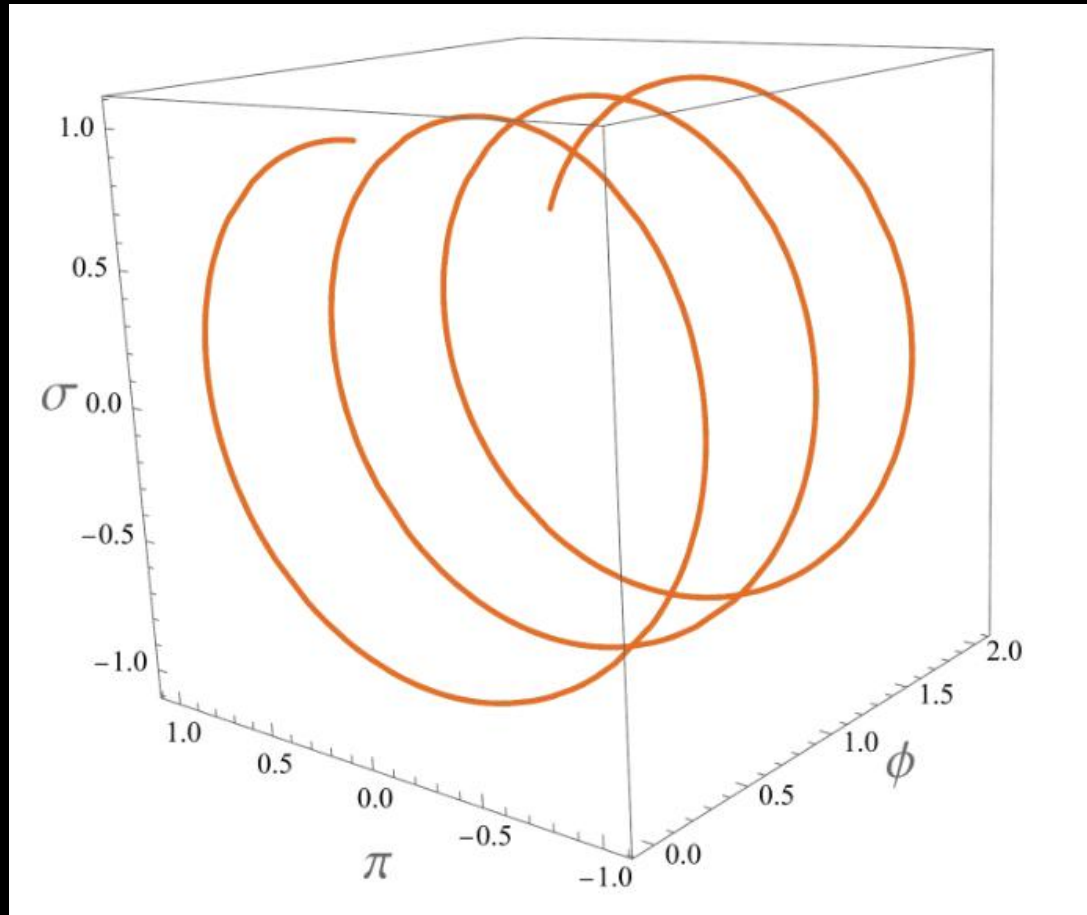
$$\sigma = M \cos(\phi), \quad \pi = M \sin(\phi).$$

Why Spirals?

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Why Spirals?



We choose the following form for ϕ ,

$$\phi = \frac{\mathbf{q} \cdot \mathbf{x}}{\hbar}$$

\mathbf{q} is called the inhomogeneity factor

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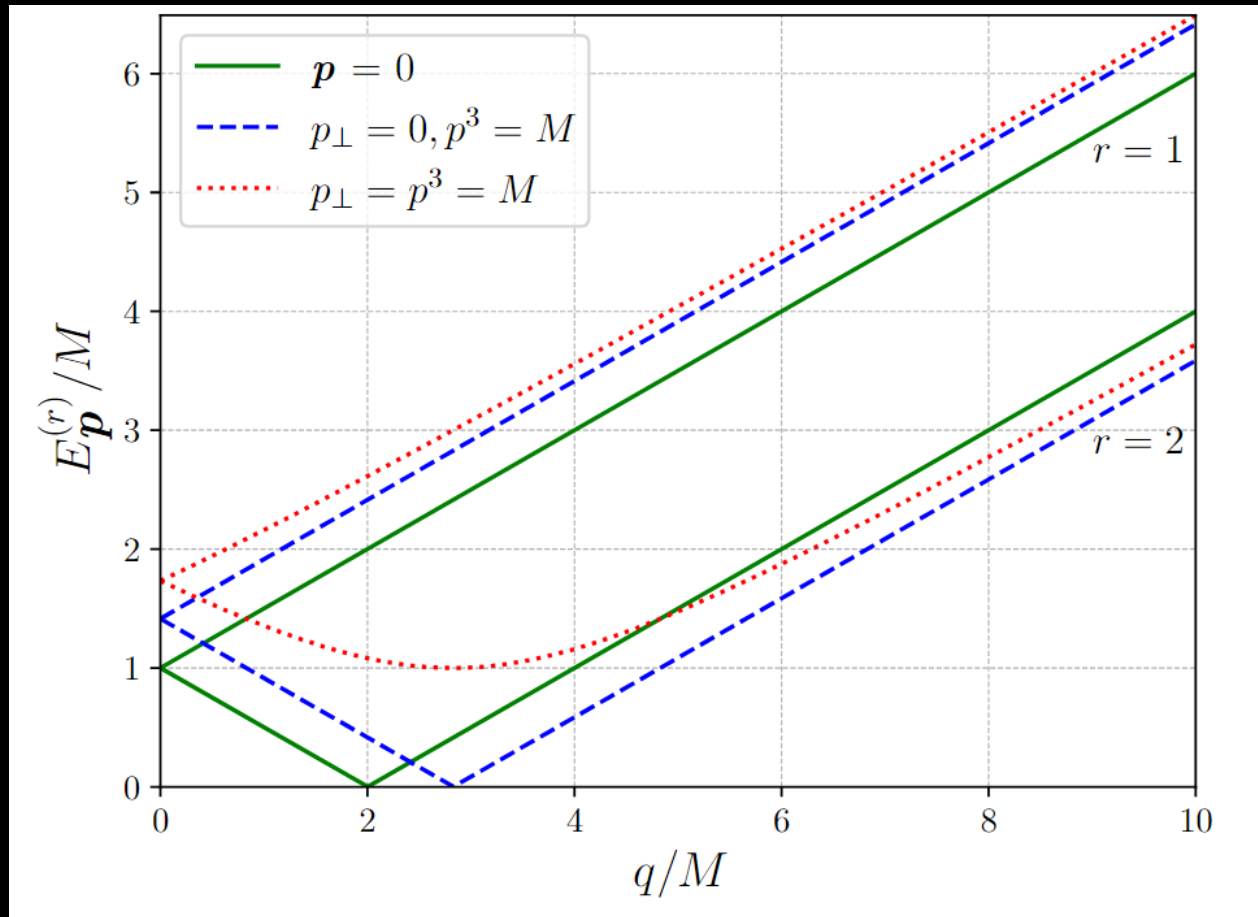
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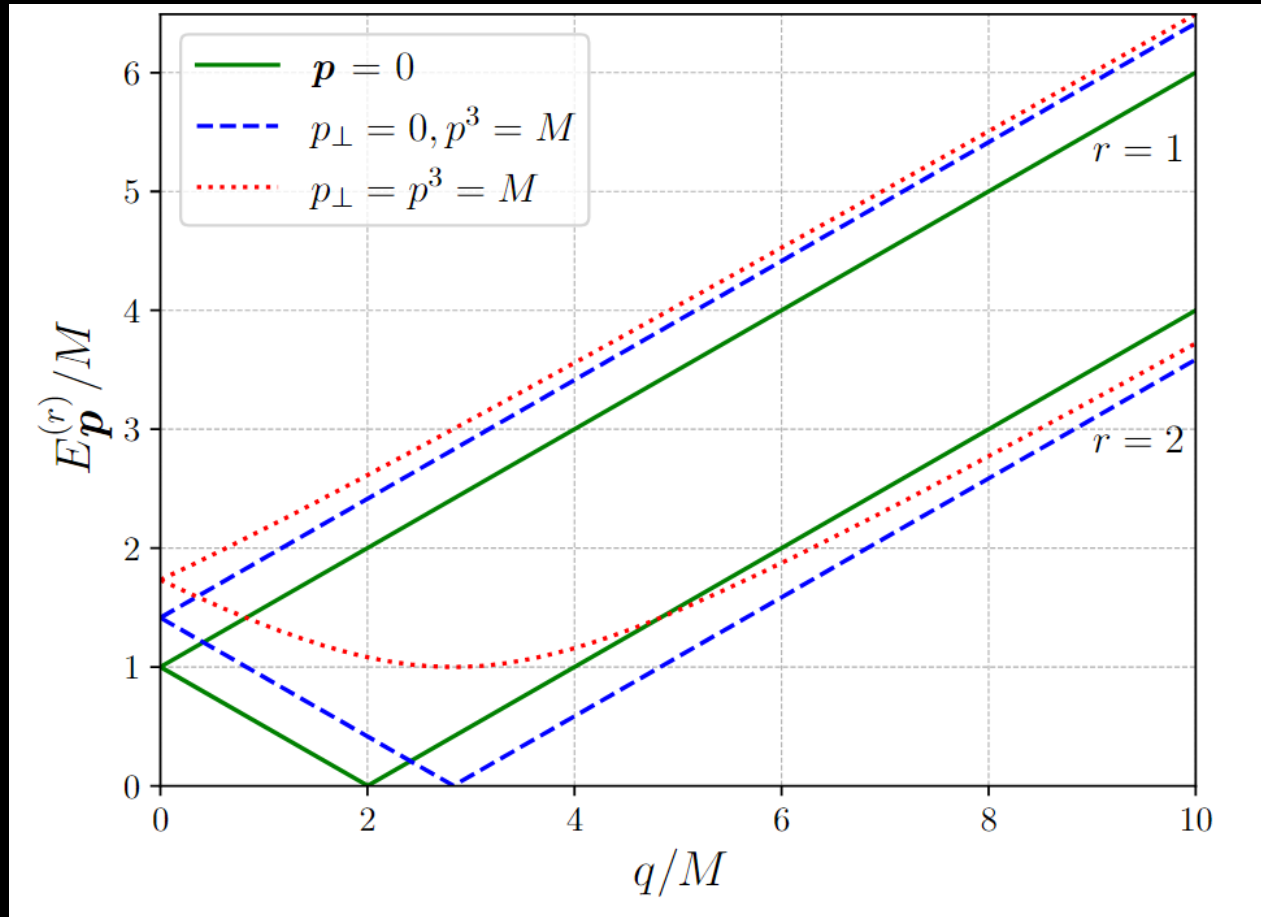
Where, $E_{\mathbf{p}}^{\parallel} = \sqrt{M^2 + (p^3)^2}$

Energy spectrum of the system

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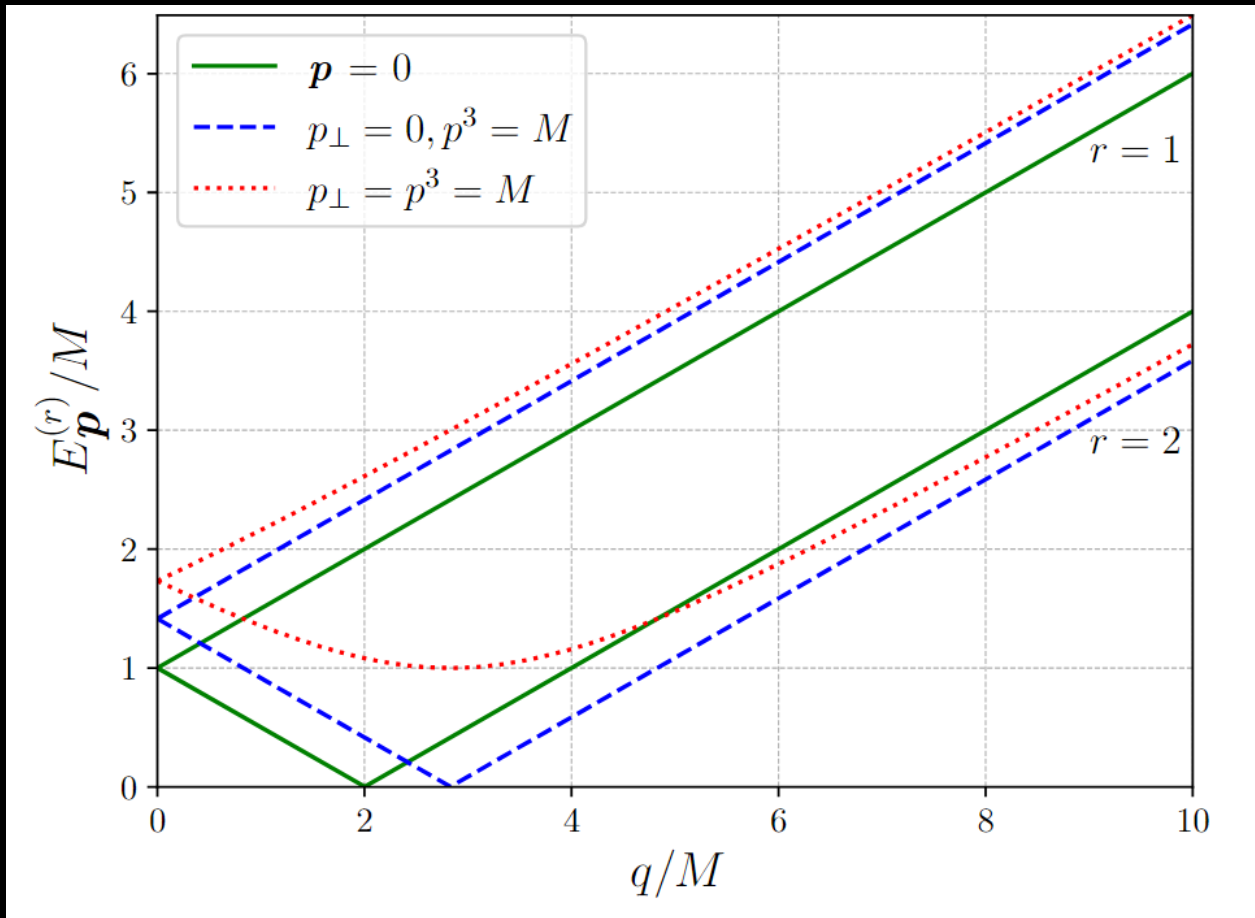


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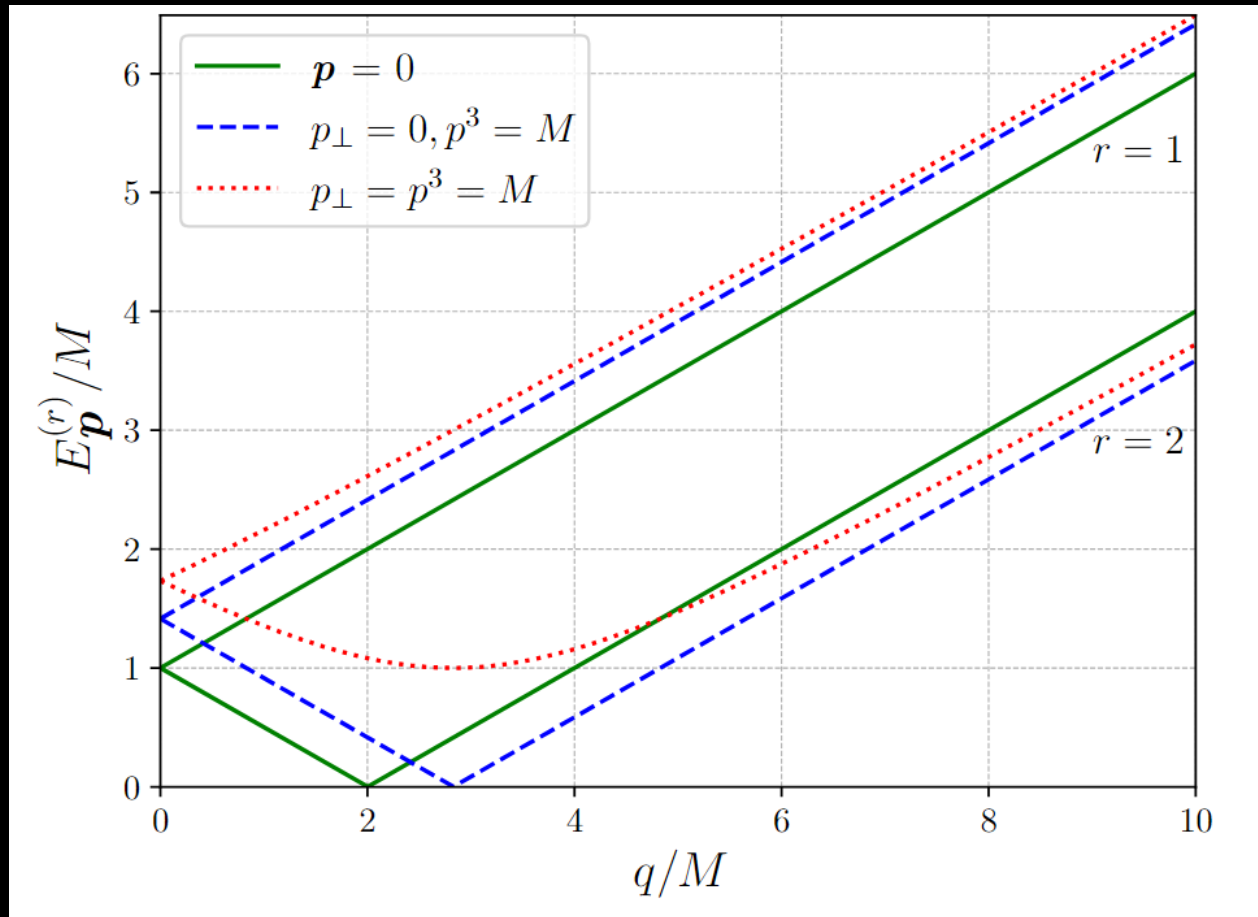


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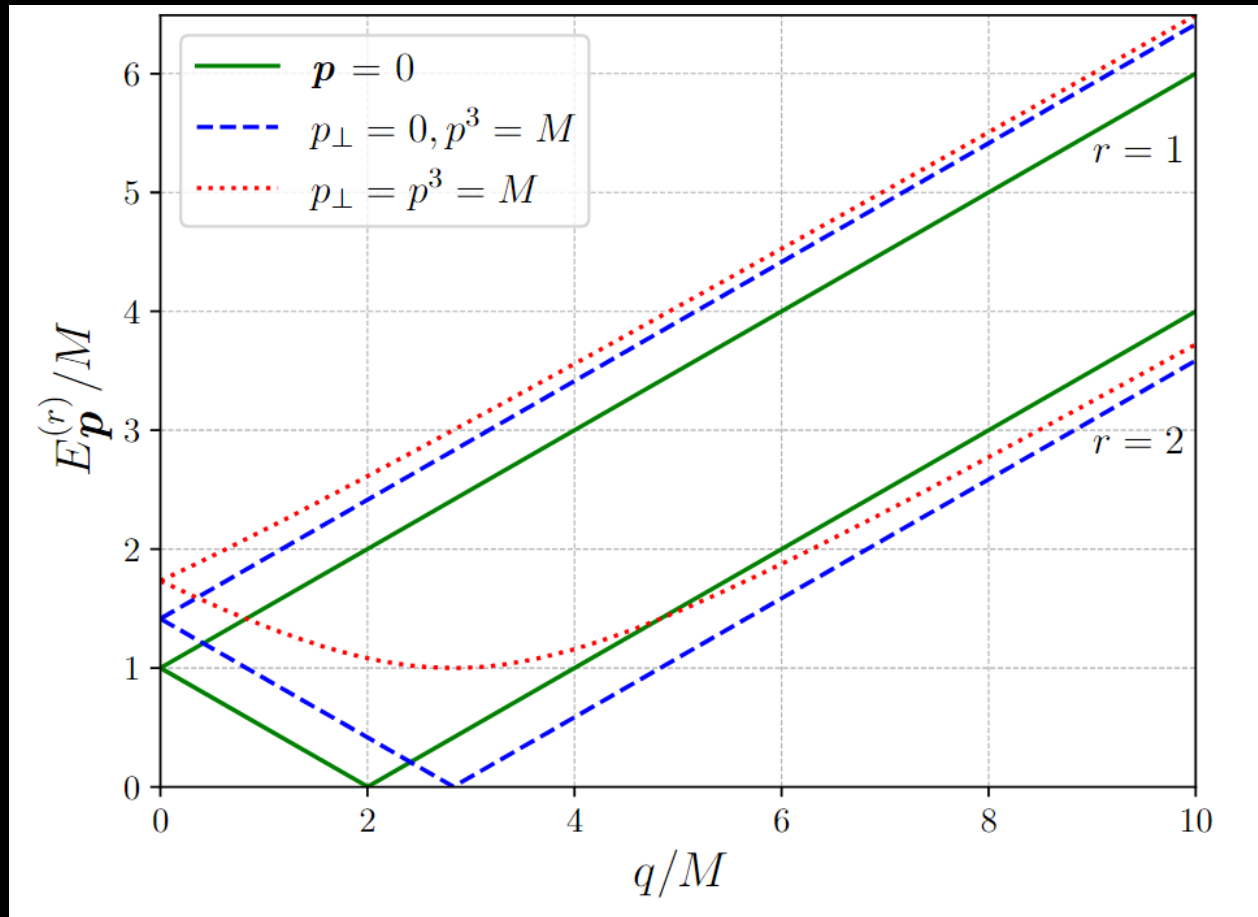


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Spinors



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$$\chi_{\pm}^{(r)}(\mathbf{p}) = N_{\pm}^{(r)} \begin{pmatrix} \frac{E_{\mathbf{p}}^{(r)} \pm (-1)^r E_{\mathbf{p}}^{\parallel} \mp \frac{q}{2}}{p^1 + ip^2} \\ \pm \frac{p^3}{M + (-1)^r E_{\mathbf{p}}^{\parallel}} \\ \pm \frac{E_{\mathbf{p}}^{(r)} \pm (-1)^r E_{\mathbf{p}}^{\parallel} \mp \frac{q}{2}}{p^1 + ip^2} \frac{p^3}{M + (-1)^r E_{\mathbf{p}}^{\parallel}} \\ 1 \end{pmatrix}$$

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The completeness relation reads

$$\sum_{r=1}^2 \frac{1}{2E_{\mathbf{p}}^{(r)}} \left[\chi_{+,a}^{(r)}(\mathbf{p})\chi_{+,b}^{(r)\dagger}(\mathbf{p}) + \chi_{-,a}^{(r)}(-\mathbf{p})\chi_{-,b}^{(r)\dagger}(-\mathbf{p}) \right] = \delta_{ab}.$$

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<p>“Up”</p> $\chi_+^{(2)}(\mathbf{p} \rightarrow 0) = e^{-i\phi} \sqrt{ 2M - q } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \uparrow$	<p>“Down”</p> $\chi_+^{(1)}(\mathbf{p} \rightarrow 0) = -\text{sgn}(\cos\theta) \sqrt{2M + q} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \downarrow$
<p>“Up”</p> $\chi_-^{(2)}(\mathbf{p} \rightarrow 0) = \sqrt{ 2M - q } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \uparrow$	<p>“Down”</p> $\chi_-^{(1)}(\mathbf{p} \rightarrow 0) = e^{-i\phi} \text{sgn}(\cos\theta) \sqrt{2M + q} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \downarrow$

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Axial current polarization

Clifford algebra decomposition of the Wigner function

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Gamma
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Gamma matrices \longrightarrow Clifford algebra
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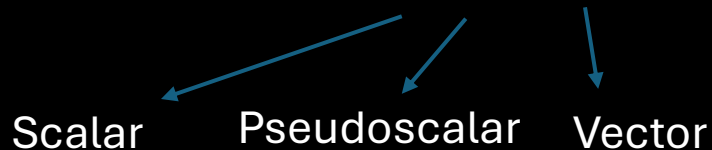
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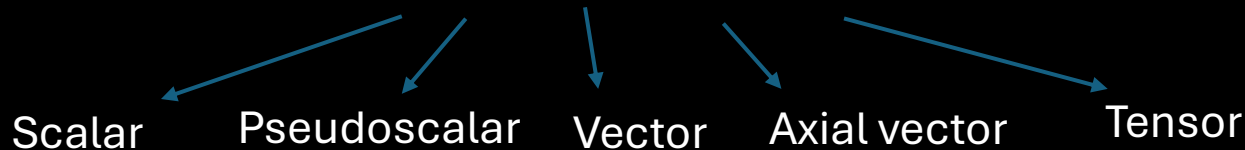
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- Currents \rightarrow Integrate the components over 4-momentum

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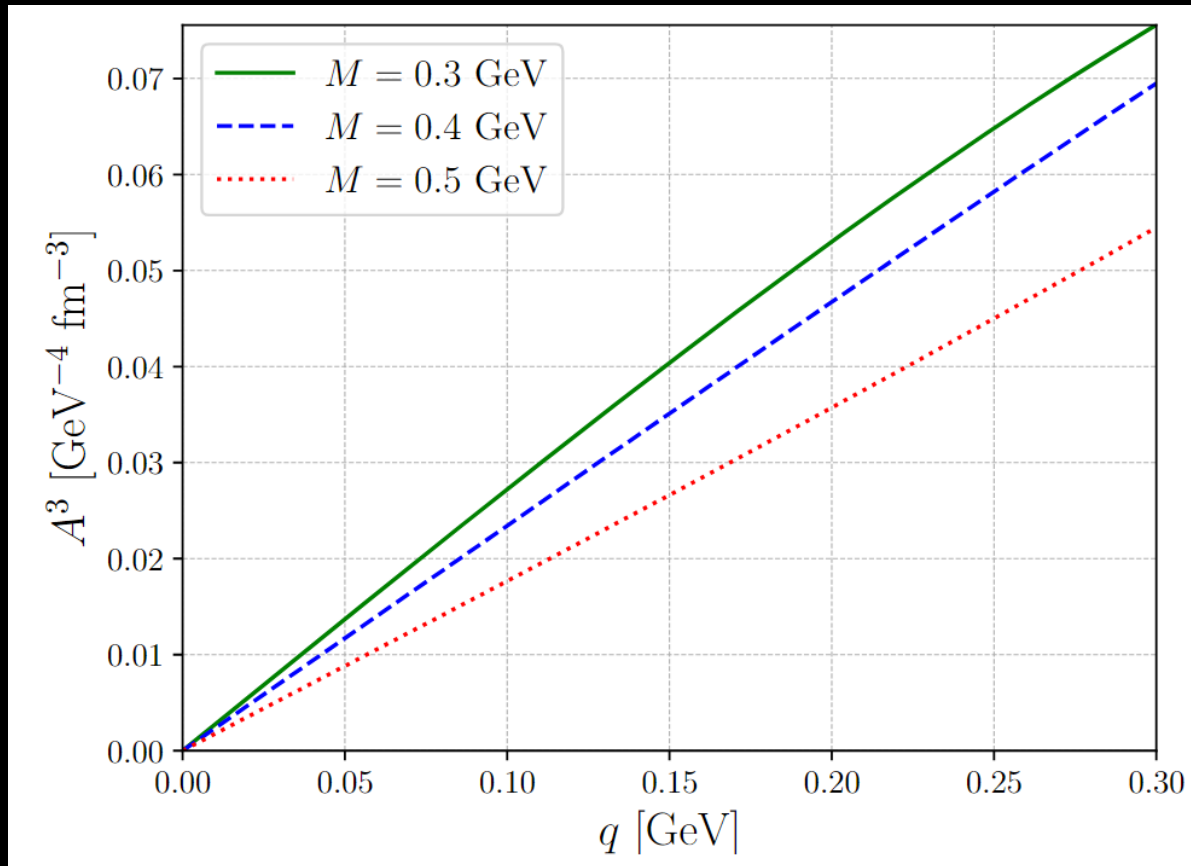
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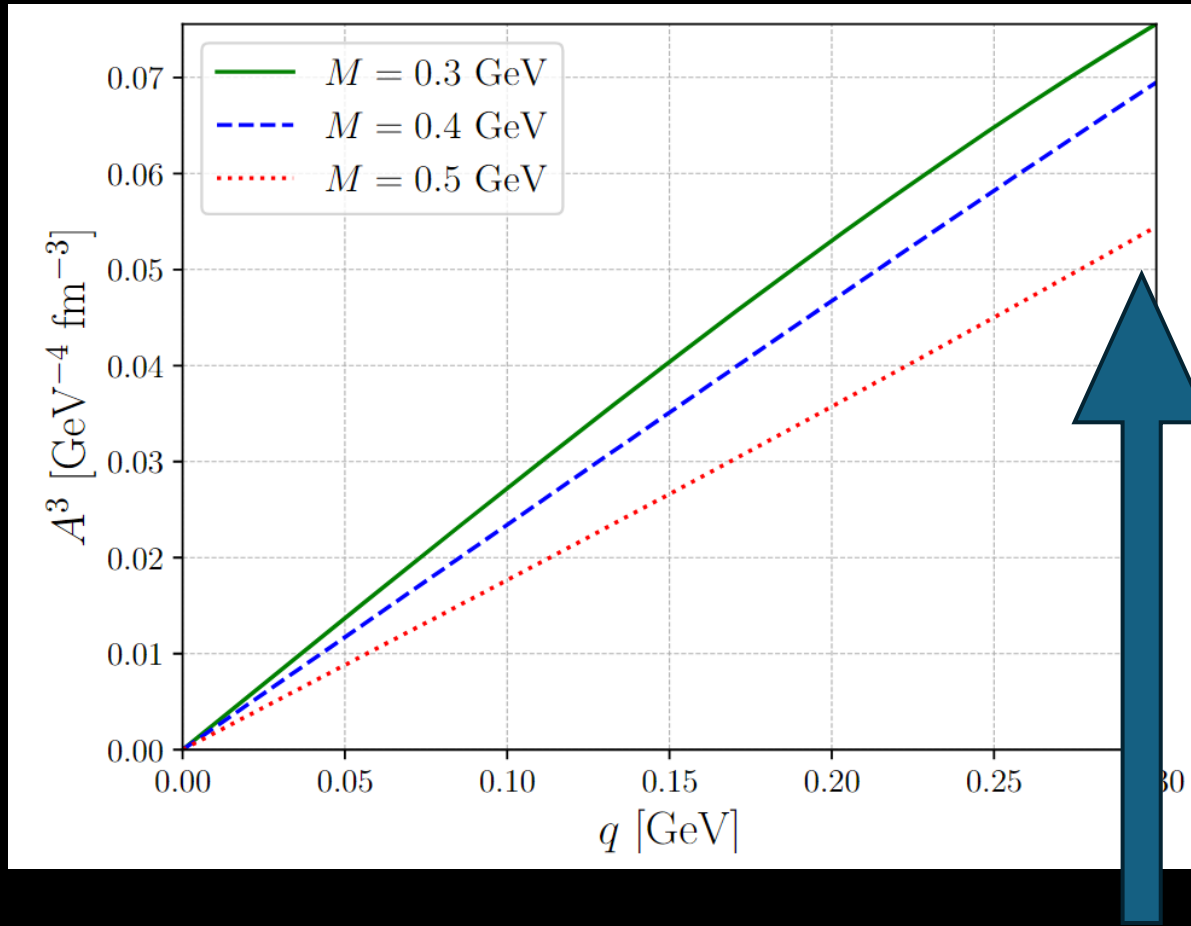
- This shows good agreement with the formula obtained in [\[M. Kutschera, W. Broniowski, and A. Kotlorz, Nuclear Physics A 516, 566 \(1990\).\]](#) up to the internal degrees of freedom connected with flavor and color.

Effects of inhomogeneity on axial current

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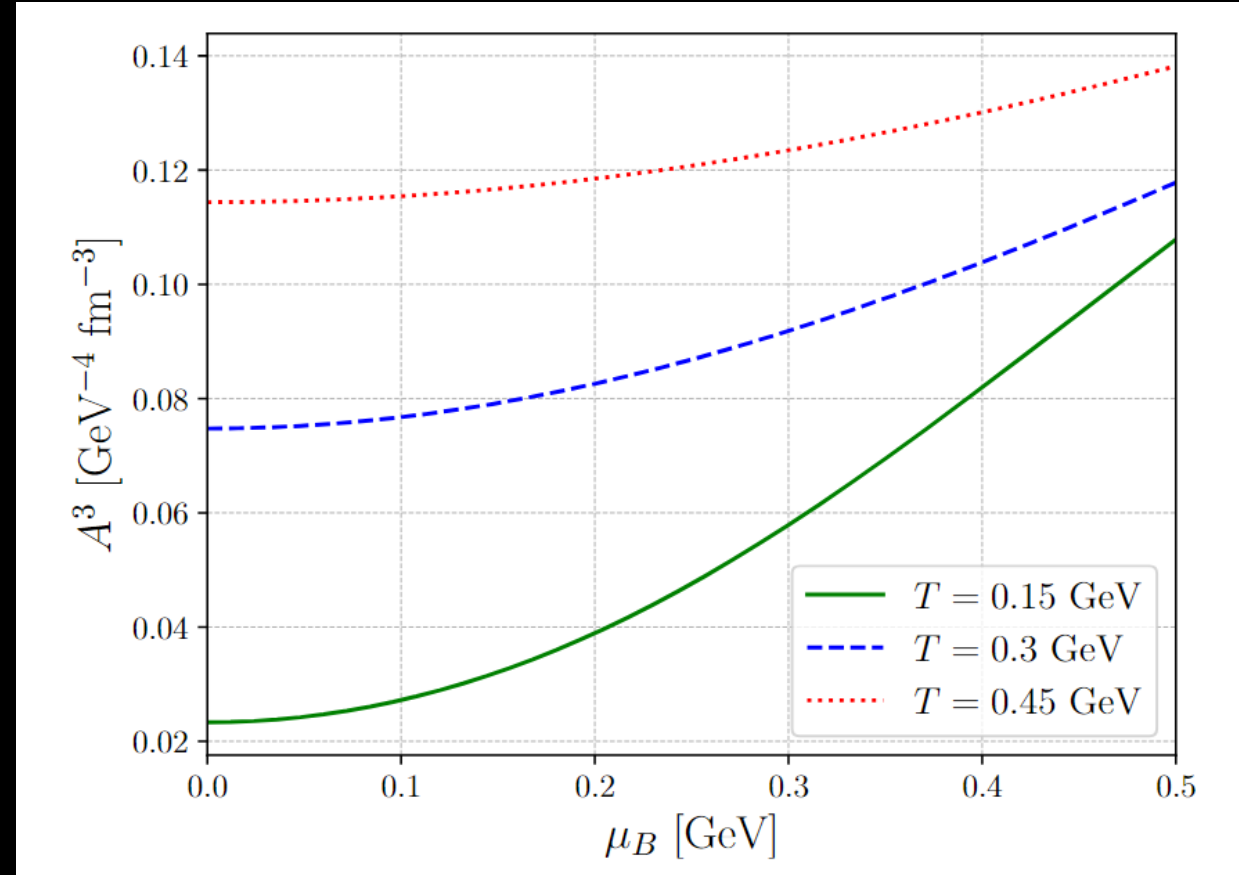
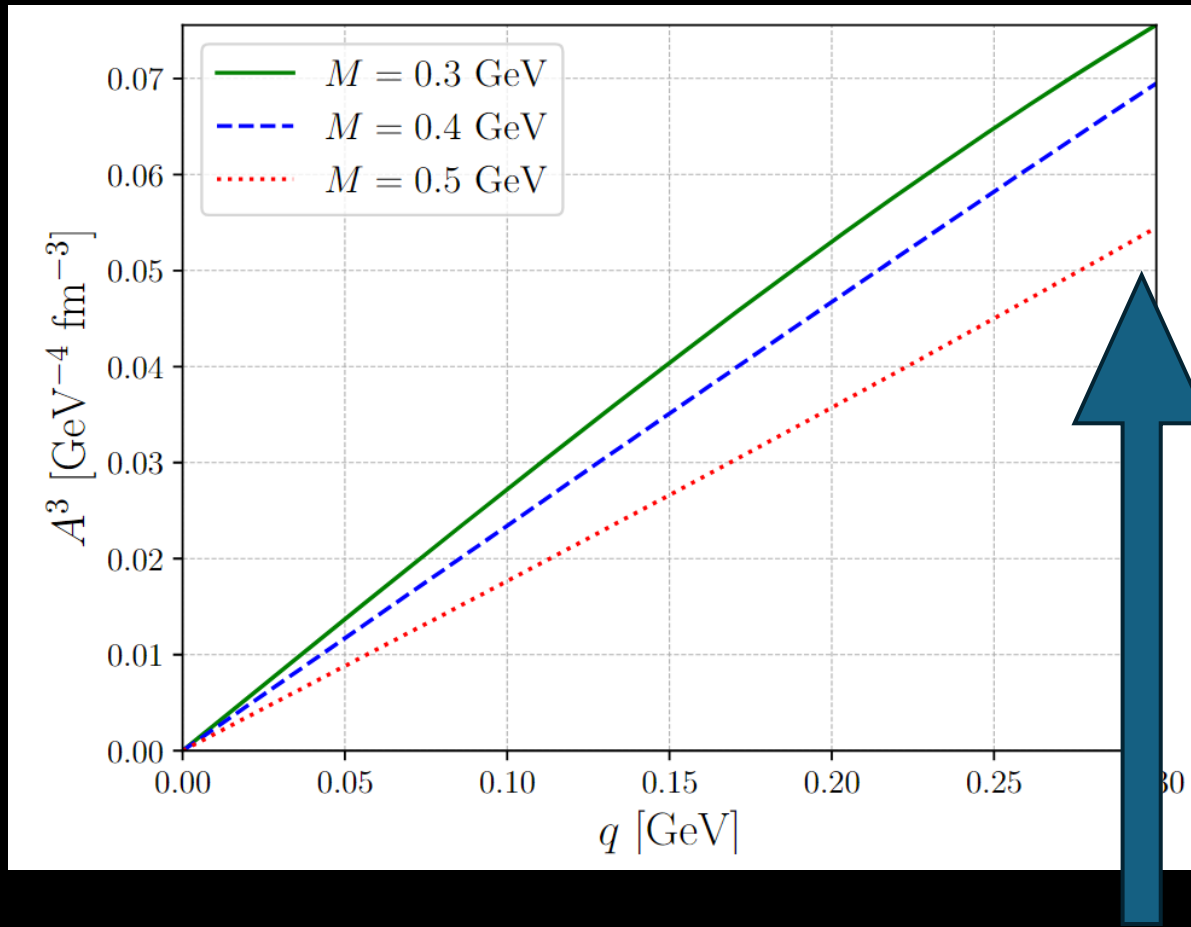


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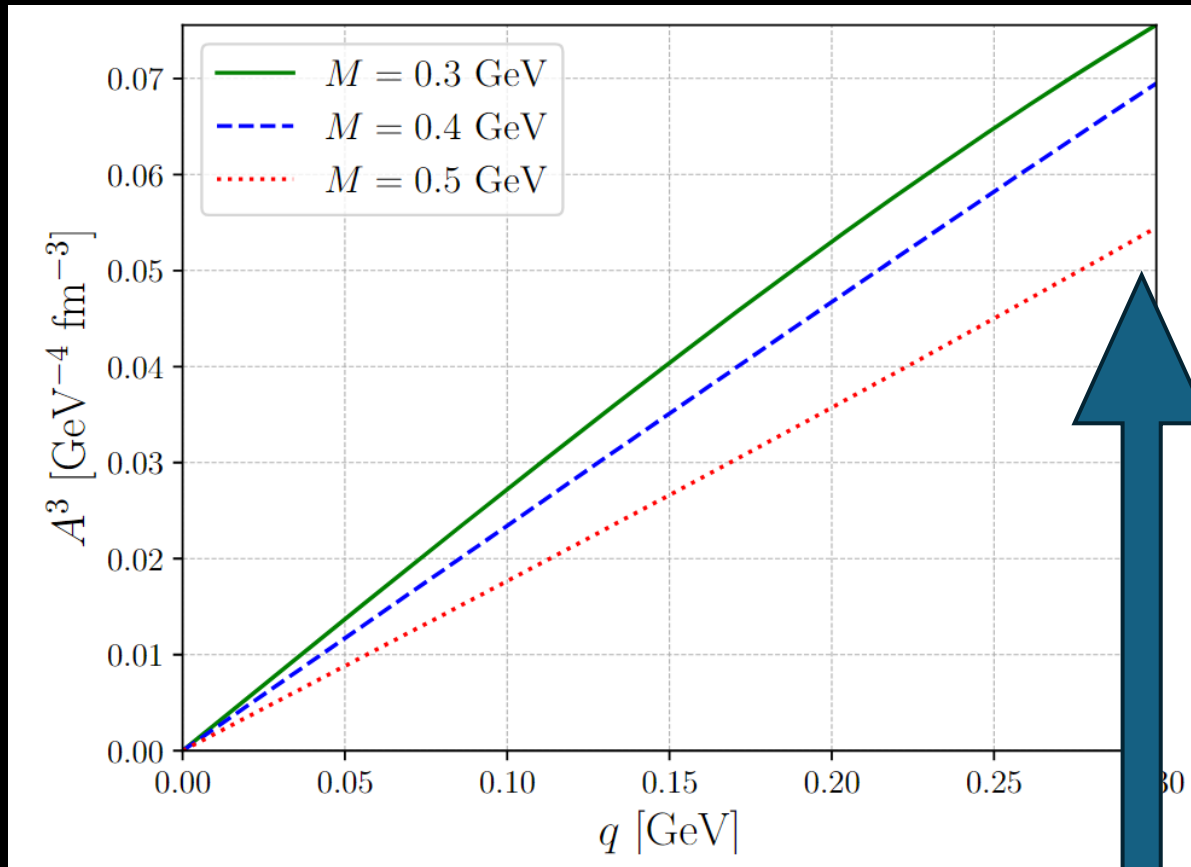
More inhomogeneity = More polarization

Effects of inhomogeneity on axial current

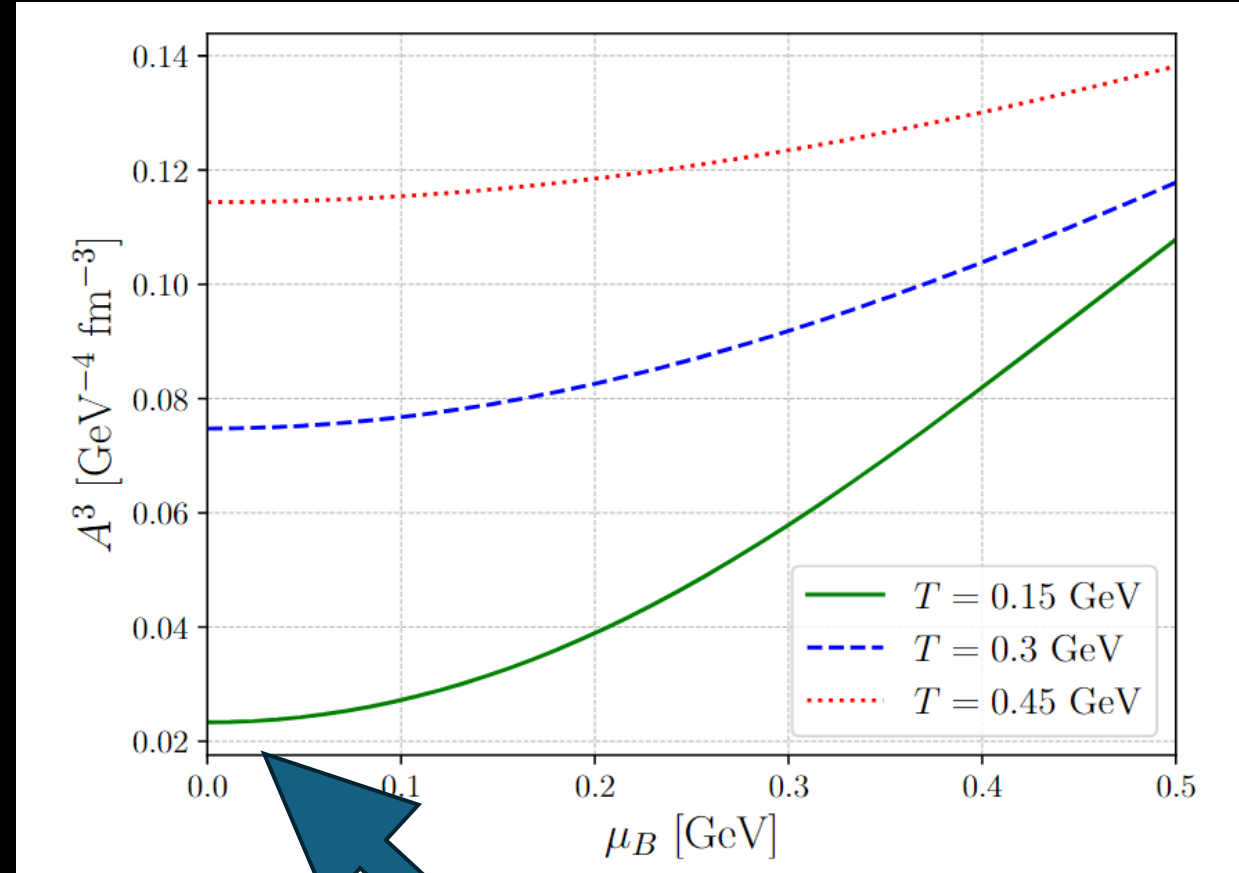


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Polarization even at vanishing baryon chemical potential

The homogeneous limit

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Expected effect



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 Put $k = (k^0, 0)$
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Thank You!

Appendix

$$k^0 \mathcal{V}^0(k^0, \mathbf{k} = 0) = \sum_{r=1}^2 k^0 \left\{ \left[\delta\left(k^0 - E_{\frac{q}{2}}^{(r)}\right) F_{\frac{q}{2}}^{(r)} - \delta\left(k^0 + E_{\frac{q}{2}}^{(r)}\right) \left(\bar{F}_{\frac{q}{2}}^{(r)} - 1\right) \right] \frac{C_r\left(\frac{q}{2}, \frac{q}{2}\right)}{2} \right. \\ \left. + \left[\delta\left(k^0 - E_{\frac{q}{2}}^{(r)}\right) F_{\frac{q}{2}}^{(r)} - \delta\left(k^0 + E_{\frac{q}{2}}^{(r)}\right) \left(\bar{F}_{\frac{q}{2}}^{(r)} - 1\right) \right] \frac{C_{r-1}\left(\frac{q}{2}, \frac{q}{2}\right)}{2} \right\} \quad (83)$$

and

$$\mathcal{M}(k^0, \mathbf{k} = 0) = \sum_{r=1}^2 \frac{M^2}{E_0^{(r)}} \left[F_0^{(r)} \delta\left(k^0 - E_0^{(r)}\right) + \left(\bar{F}_0^{(r)} - 1\right) \delta\left(k^0 + E_0^{(r)}\right) \right] C_r\left(0, \frac{q}{2}\right). \quad (84)$$