SPIN HYDRODYNAMICS

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Strange and heavy Flavour Physics
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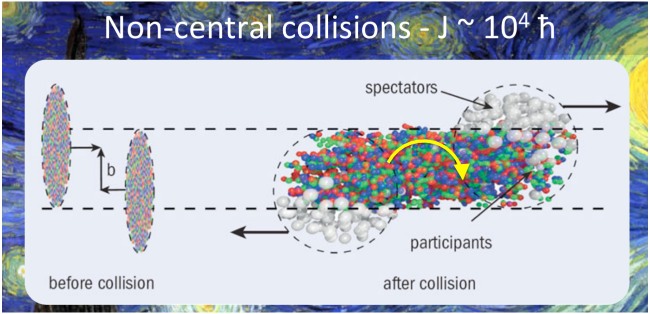
PART 1: PHYSICS MOTIVATION

Non-central heavy-ion collisions create fireballs with large global angular momenta, some part of the angular momentum can be transferred from the orbital to the spin part

$$\boldsymbol{J}_{\rm init} = \boldsymbol{L}_{\rm init} = \boldsymbol{L}_{\rm final} + \boldsymbol{S}_{\rm final}$$



(Michael Lisa, talk „Strangeness in Quark Matter 2016")

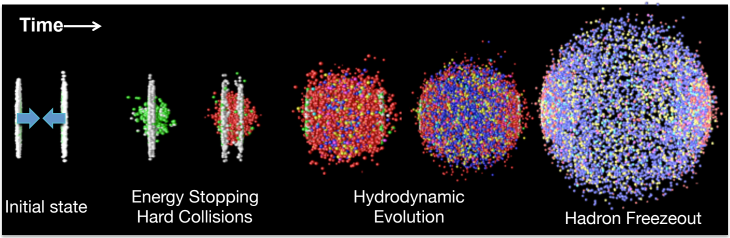
Warning: large angular momentum does not mean large angle of rotation!

$\Delta t = 1$ fm/c = 3×10^{-24} s, $\quad \Delta\phi = \Delta t\, \omega_{\rm max} = 27 \times 10^{-24} \times 10^{21} = 2.7 \times 10^{-2}$

PART 2: STANDARD RELATIVISTIC HYDRODYNAMIC

Standard model (scheme) of heavy-ion collisions

RELATIVISTIC HYDRODYNAMICS FORMS THE BASIC INGREDIENT OF THE STANDARD MODEL OF HEAVY-ION COLLISIONS



T. K. Nayak, Lepton-Photon 2011 Conference

data on spin polarization suggest that spin should be included in the hydrodynamic framework

Perfect fluid hydrodynamics

PERFECT-FLUID HYDRODYNAMICS = local equilibrium + conservation laws

one usually includes energy, linear momentum, baryon number, ...

T (temperature), u^μ (three independent components of flow), μ (baryon chemical potential)
ε (energy density), P (pressure), n (baryon density), σ (entropy density), $\xi = \mu/T$

$$T^{\mu\nu} = [\varepsilon(T,\mu) + P(T,\mu)]\, u^\mu u^\nu - P(T,\mu) g^{\mu\nu} \qquad (1)$$

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu N^\mu = \partial_\mu (n u^\mu) = 0 \qquad \text{(4+1 eqs.)} \qquad (2)$$

five equations for five unknown functions

dissipation does not appear

$$\partial_\mu S^\mu = \partial_\mu (\sigma u^\mu) = 0 \qquad \text{(1 eq.)} \qquad (3)$$

entropy conservation follows from the energy-momentum conservation and the form of the energy-momentum tensor
Euler's equation (says that four-acceleration is caused by the pressure gradient)

$$u^\lambda \partial_\lambda u^\mu \equiv a^\mu = \frac{(g^{\mu\nu} - u^\mu u^\nu)}{\varepsilon + P} \partial_\nu P \equiv \frac{1}{\varepsilon + P} \Delta^{\mu\nu} \partial_\nu P \qquad \text{(3 eqs.)} \qquad (4)$$

PART 3: CLASSICAL APPROACH TO SPIN

A particle can be characterised by the internal angular momentum tensor $s^{\alpha\beta}$
M. Matthison, Acta Phys. Polon. 6 (1937) 163

Neue Mechanik materieller Systeme

Nowa mechanika systemów materialnych

Von MYRON MATHISSON, Warschau

(Eingegangen am 8. September 1937)

Mathisson with Pauli
Copenhagen 1937

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta} p_\gamma s_\delta, \quad s\cdot p = 0, \quad s^\alpha = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta} p_\beta s_{\gamma\delta} \qquad (5)$$

A straightforward generalization of the phase-space distribution function $f(x, \boldsymbol{p})$ is a spin dependent distribution $f(x, \boldsymbol{p}, s)$
WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

$$\int dS \ldots = \frac{m}{\pi \mathfrak{s}} \int d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s) \ldots \qquad \mathfrak{s}^2 = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4} \qquad (6)$$

$$\int dS = \frac{m}{\pi \mathfrak{s}} \int d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s) = 2 \qquad (7)$$

from now on, we consider spin 1/2 only

Local equilibrium distribution without spin: Maxwell-Jüttner distribution

Maxwell distribution

$$f_{\rm eq}(\boldsymbol{v}) = \left[\frac{m}{2\pi k_B T}\right]^{3/2} \exp\left[-\frac{m\boldsymbol{v}^2}{2k_B T}\right] \tag{8}$$

Maxwell-Jüttner distribution (natural units, Boltzmann statistics)

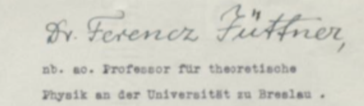
$$f_{\rm eq}(\boldsymbol{p}) = \exp\left[-\frac{\sqrt{m^2+\boldsymbol{p}^2}}{T}\right] \to \exp\left[-\frac{\sqrt{m^2+\boldsymbol{p}^2}-\mu}{T}\right] \tag{9}$$

$f(x,\boldsymbol{p})$ phase space distribution for unpolarized systems, Lorentz scalar

$$f_{\rm eq}(x,p) = 2\exp\left[-\frac{p^\mu u_\mu(x)-\mu(x)}{T(x)}\right] = 2\exp\left[-p^\mu \beta_\mu(x) + \xi(x)\right] \tag{10}$$

$\xi = \mu/T$ ratio of the baryon chemical potential and temperature, $\beta^\mu = u^\mu/T$ ratio of the hydrodynamic flow and temperature, 2 - spin degeneracy

$u'^\mu = (1,0,0,0)$ **in the local fluid rest frame (LRF)**



WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

spin conserving equilibrium distribution functions for particles and antiparticles

$$f^{\pm}_{\rm eq}(x,p,s) = \exp\left(\pm\xi(x) - p\cdot\beta(x) + \tfrac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right) \tag{11}$$

macroscopic currents

baryon current

$$N^{\lambda}_{\rm eq} = \int dP \int dS \, p^{\lambda} \, 1 \left[f^{+}_{\rm eq}(x,p,s) - f^{-}_{\rm eq}(x,p,s)\right] \tag{12}$$

energy-momentum tensor

$$T^{\lambda\mu}_{\rm eq} = \int dP \int dS \, p^{\lambda} p^{\mu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s)\right] \tag{13}$$

spin tensor

$$S^{\lambda,\mu\nu}_{\rm eq} = \int dP \int dS \, p^{\lambda} \, s^{\mu\nu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s)\right] \tag{14}$$

construction of the equilibrium function implies the conservation laws

$$\partial_\mu N^\mu(x) = 0, \quad \partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}(x) = 0 \qquad (15)$$

these are 11 equations for 11 Lagrange multipliers

$$\partial_\mu N^\mu[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(1 eq.)} \qquad (16)$$

$$\partial_\mu T^{\mu\nu}[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(4 eqs.)} \qquad (17)$$

$$\partial_\lambda S^{\lambda,\mu\nu}[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(6 eqs.)} \qquad (18)$$

PERFECT-SPIN HYDRODYNAMICS

WF, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C97 (2018) 041901

CLASSICAL SPIN ≡ SPIN OPERATOR EXPECTATION VALUE

classical spin should be undestood as the expectation value of the spin polarization operator

VOLUME 2, NUMBER 10 PHYSICAL REVIEW LETTERS MAY 15, 1959

PRECESSION OF THE POLARIZATION OF PARTICLES MOVING IN A HOMOGENEOUS ELECTROMAGNETIC FIELD*

V. Bargmann
Princeton University, Princeton, New Jersey
Louis Michel
Ecole Polytechnique, Paris, France
and
V. L. Telegdi
University of Chicago, Chicago, Illinois
(Received April 27, 1959)

The problem of the precession of the "spin" of a particle moving in a homogeneous electromagnetic field – a problem which has recently acquired considerable experimental interest – has already been investigated for spin $\frac{1}{2}$ particles in some particular cases.[1] In the literature the results were derived by explicit use of the Dirac equation, with the occasional inclusion of a Pauli term to account for an anomalous magnetic moment. On the other hand, following a remark of Bloch[2] in connection with the nonrelativistic case, the expectation value of the vector operator representing the "spin" will necessarily follow the same time dependence as one would obtain from a classical equation of motion. To solve the pro-

customary equation of motion

$$d\vec{s}/d\tau = (ge/2m)(\vec{s}\times\vec{H}), \qquad (R) \qquad (3)$$

where $\vec{H}$, e, and m have their standard meanings, while the gyromagnetic ratio g is defined by this very equation. While s^0 vanishes by hypothesis in any instantaneous rest-frame, $ds^0/d\tau$ need not. In fact, (2) implies

$$ds^0/d\tau = \vec{s}\cdot(d\vec{v}/d\tau), \qquad (R) \qquad (4)$$

for such frames. In general, $du/d\tau = f/m$ (where f = four-force), while in a homogenous external electromagnetic field specified by $F = -(\vec{E}, \vec{H})$

$$du/d\tau = (e/m)F\cdot u. \qquad (5)$$

CONSERVED SPIN – PHYSICALLY MOTIVATED STARTING POINT

standard spin-orbit (as considered in atomic physics) is mediated by the magnetic field, coherent process, can be included in spin magneto-hydrodynamics (spin-MHD)
S. Bhadury, WF, A. Jaiswal, A. Kumar, R. Ryblewski, PRL129 (2022) 192301

conservation of the total angular momentum of a particle implies conservation of spin if collisions are local (spacetime coordinate x^μ can be always set equal to 0)

$$j^{\alpha\beta} = l^{\alpha\beta} + s^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha + s^{\alpha\beta} \tag{19}$$

transfer between orbital and spin part is possible if collisions are non-local , this leads to dissipation and entropy production, **series of influential works by the Frankfurt group**
N. Weickgenannt, E. Speranza, Xin-Li Sheng, Q. Wang, D. Rischke, PRL127 (2021) 052301

perfect spin hydrodynamics = spin conservation

dissipative spin hydrodynamics = transfer between ***S*** and ***L*** possible

Joseph Kapusta: strange quark spin or helicity is unchanged from the time they are created to the time they hadronize
J. Kapusta, E. Rrapaj, S. Rudaz, PRC101 (2020) 024907
Sidney Coleman's QFT: as the collision energies decrease, all processes are dominated by s-wave scattering

perfect spin hydro (conserving spin) is a convenient starting point to construct the formalism

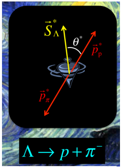
PART 3: QUANTUM APPROACH TO SPIN

Quantum spin description: spin vs. spinor density matrix

standard scalar functions $f(x,p)$ are generalized to **2×2 Hermitian matrices** in spin space for each value of the space-time position x and four-momentum p, the sign ± distinguishes particles from antiparticles, $\boldsymbol{\sigma}$ - Pauli matrices

$\zeta^\pm_* = 0$ no polarization, $\zeta^\pm_* = 1$ pure state, $0 < |\zeta^\pm_*| < 1$ mixed state, **asterisk denotes the particle rest frame (PRF)**

$$f^\pm_{rs}(x,p) = f^\pm_0(x,p)\left[\delta_{rs} + \boldsymbol{\zeta}^\pm_*(x,\boldsymbol{p})\cdot\boldsymbol{\sigma}_{rs}\right], \quad 0 \le |\boldsymbol{\zeta}^\pm_*| \le 1 \tag{20}$$



$\boldsymbol{\zeta}^\pm_*(x,\boldsymbol{p})$ can be interpreted as a spatial part of the polarization four-vector $\zeta^{\pm\mu}_*(x,\boldsymbol{p})$ with a vanishing zeroth component

$$\zeta^{\pm\mu}_* = \left(0, \boldsymbol{\zeta}^\pm_*\right) \tag{21}$$

in the LAB frame – $\zeta^{\pm\mu}_*$ boosted with the velocity defined by the particle velocity

$$\zeta^\mu_\pm = \Lambda^\mu{}_\nu(\boldsymbol{v}_p)\zeta^\nu_{\pm *} = \left(\frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}^\pm_*}{m}, \boldsymbol{\zeta}^\pm_* + \frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}^\pm_*}{m(E_p+m)}\boldsymbol{p}\right), \qquad \zeta^\mu_\pm p_\mu = 0 \tag{22}$$

transition to **4×4 spinor density matrices** $X^\pm$

$$f^+_{rs}(x,p) \quad = \quad \bar{u}_r(p)X^+u_s(p), \qquad f^-_{rs}(x,p) = -\bar{v}_s(p)X^-v_r(p) \tag{23}$$

$u_s(p)$ and $v_r(p)$ – Dirac bispinors

Quantum spin description: local equilibrium spinor density matrix

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32
WF, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C97 (2018) 041901

$\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$ is the Dirac spin operator

$$X^\pm(x,p) = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu \pm \frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right] \tag{24}$$

$\omega_{\mu\nu} = \Omega_{\mu\nu}/T$ ratio of the tensor spin chemical potential and temperature, altogether we have 11 Lagrange multipliers that control the conservation of the baryon number (1), energy (1), linear momentum (3), angular momentum (3), and Lorentz boost vectors (3)

WF, B. Friman, A. Jaiswal, E. Speranza, R. Ryblewski, Phys. Rev. D97 (2018) 116017
problems with the normalization of the polarization vector

REVISED FORMULA: S. Bhadury, Z. Drogosz, WF, S. K. Kar, V. Mykhaylova, **arXiv:2505.02657**
with a spacelike four-vector a^μ ($a^2 < 0$)

$$X^\pm = \exp\left(\pm\xi - \beta_\mu p^\mu + \gamma_5 \not{a}\right) = \exp\left(\pm\xi - \beta_\mu p^\mu\right) \cosh\sqrt{-a^2}\left[1 + \frac{\gamma_5 \not{a}}{\sqrt{-a^2}} \tanh\sqrt{-a^2}\right] \tag{25}$$

$$a_\mu(x,p) = -\frac{1}{2m}\tilde{\omega}_{\mu\nu}(x)p^\nu \tag{26}$$

more in the next talk by Valeriya Mykhaylova

Quantum spin description: macroscopic currents

S. R. de Groot, W. A. van Leeuwen, Ch. G. van Weert, **Relativistic kinetic theory**, North-Holland Publishing Company 1980

baryon current: $$N^\lambda(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^\lambda\, 1 \left[f^+_{rr}(x,p) - f^-_{rr}(x,p) \right] \quad (27)$$

energy-momentum tensor: $$T^{\lambda\mu}(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^\lambda p^\mu \left[f^+_{rr}(x,p) + f^-_{rr}(x,p) \right] \quad (28)$$

spin tensor: $$S^{\lambda,\mu\nu}(x) = \frac{1}{2} \sum_{r,s=1}^{2} \int \mathrm{d}P\, p^\lambda \left[\sigma^{+\mu\nu}_{sr}(p) f^+_{rs}(x,p) + \sigma^{-\mu\nu}_{sr}(p) f^-_{rs}(x,p) \right] \quad (29)$$

where $\sigma^{+\mu\nu}_{sr}(p) = 1/(2m)\, \bar{u}_s(p)\sigma^{\mu\nu} u_r(p)$ and $\sigma^{-\mu\nu}_{sr}(p) = 1/(2m)\, \bar{v}_r(p)\sigma^{\mu\nu} v_s(p)$, with $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$

these forms of currents are commonly known as the GLW versions (GLW pseudogauge)
for free Dirac equation (relativistic gas) these tensors are conserved

$$\partial_\mu N^\mu(x) = 0, \quad \partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}(x) = 0 \quad (30)$$

connections between classical and quantum spin: talk by Zbigniew Drogosz

PART 3: PSEUDOGAUGE FREEDOM

vs. PSEUDOGAUGE DEPENDENCE

BLASPHEMY #3 (GLW form): ENERGY-MOMENTUM TENSOR SHOULD BE USED IN THE BELINFANTE-ROSENFELD FORM, SINCE THIS FORM APPEARS IN EINSTEIN´S EQUATIONS

Pseudo-gauge transformation (QCD language in the context of the proton spin puzzle: adding boundary terms)

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\Phi^{\lambda,\mu\nu} + \Phi^{\nu,\mu\lambda} + \Phi^{\mu,\nu\lambda}\right) \tag{31}$$

$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\rho Z^{\mu\nu,\lambda\rho} \tag{32}$$

Canonical forms (directly obtained from Noether's Theorem): asymmetric energy-momentum tensor, spin tensor directly expressed by axial current (couples to weak interactions)

Belinfante-Rosenberg version, $\Phi^{\lambda,\mu\nu} = S^{\lambda,\mu\nu}, Z^{\mu\nu,\lambda\rho} = 0$, (couples to classical gravity); spin tensor appears in modified theories of gravity, couples to torsion

de Groot, van Leuveen, van Weert (GLW) forms: symmetric energy-momentum tensor and conserved spin tensor

Hilgevoord and Wouthuysen (HW) choice: symmetric energy-momentum tensor and conserved spin tensor

there is ongoing discussion if the physics is or is not pseudogauge dependent F. Hehl, Rept. Math. Phys. 9 (1976) 55

Sidney Coleman's old answer: ...we have an infinite family of possible definitions of the local current...some textbooks try to avoid this point, or nervously rub one foot across the other leg and natter about the best definition or the optimum definition...and the right answer is, of course, there's nothing to natter about, there's nothing to be disturbed about...it is something to be pleased about. If we have many objects that satisfy desirable general criteria, then that's better than having just one... the more freedom you have, the better. It's like being passed a plate of cookies and someone starts arguing about which is the best cookie. They're all edible!

WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

GLW → canonical: superpotential defined as $\Phi^{\lambda,\mu\nu}_{\rm can} \equiv S^{\mu,\lambda\nu}_{\rm GLW} - S^{\nu,\lambda\mu}_{\rm GLW}$, then we have

$$S^{\lambda,\mu\nu}_{\rm can} = S^{\lambda,\mu\nu}_{\rm GLW} - \Phi^{\lambda,\mu\nu}_{\rm can}$$

and

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu}_{\rm can} + \Phi^{\mu,\nu\lambda}_{\rm can} + \Phi^{\nu,\mu\lambda}_{\rm can}\right)$$

canonical → Belinfante-Rosenfeld: superpotential defined as $\Phi_{\rm Bel} = S^{\lambda,\mu\nu}_{\rm can}$

$$S^{\lambda,\mu\nu}_{\rm Bel} = 0, \quad T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm can} + \frac{1}{2}\partial_\lambda \left(S^{\lambda,\mu\nu}_{\rm can} - S^{\mu,\lambda\nu}_{\rm can} - S^{\nu,\lambda\mu}_{\rm can}\right),$$

PART 4: THERMODYNAMICS WITH SPIN

inclusion of spin, $\Omega_{\alpha\beta}$ - spin chemical potential

$S^{\alpha\beta}$ - spin density tensor introduced in J. Weyssenhoff, A. Raabe, Acta Phys. Polon. 9 (1947) 7

$$\varepsilon + P = T\sigma + \mu n + \tfrac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta} \tag{33}$$

$$d\varepsilon = Td\sigma + \mu dn + \tfrac{1}{2}\Omega_{\alpha\beta}dS^{\alpha\beta} \qquad dP = \sigma dT + nd\mu + \tfrac{1}{2}S^{\alpha\beta}d\Omega_{\alpha\beta} \tag{34}$$

multiplication of the above equations by the hydrodynamic flow vector u gives **the tensor (Israel-Stewart) form**
W. Israel, J.M.Stewart, Annals Phys. 118 (1979) 341 & Phys.Lett. A58 (1976) 213

$$S^{\mu}_{\rm eq} = P\beta^{\mu} - \xi N^{\mu}_{\rm eq} + \beta_{\lambda}T^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} \tag{35}$$

$$dS^{\mu}_{\rm eq} = -\xi dN^{\mu}_{\rm eq} + \beta_{\lambda}dT^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq}, \quad d(P\beta^{\mu}) = N^{\mu}_{\rm eq}d\xi - T^{\lambda\mu}_{\rm eq}d\beta_{\lambda} + \tfrac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta} \tag{36}$$

spin tensor

$$S^{\mu,\alpha\beta}_{\rm eq} = u^{\mu}S^{\alpha\beta}_{\rm eq} \tag{37}$$

analog to the perfect-fluid forms of $N^{\mu}_{\rm eq}$ and $T^{\lambda\mu}_{\rm eq}$, however, in kinetic theory we find

$$S^{\mu,\alpha\beta}_{\rm eq} = u^{\mu}S^{\alpha\beta}_{\rm eq} + \text{problem} \tag{38}$$

problem = term that is not proportional to u^{λ}

a solution stands behind the corner ....

WF, M. Hontarenko, PRL134 (2025) 082302
Z.. Drogosz, WF, M. Hontarenko, PRD 110 (2024) 096018
Boltzmann's definition of the entropy (H-function)

$$S^\mu = -\int dP\, dS\, p^\mu \left[f^+ \left(\ln f^+ - 1\right) + f^- \left(\ln f^- - 1\right)\right] \quad \text{classical spin} \tag{39}$$

$$S^\mu = -\frac{1}{2} \int \mathrm{d}P\, p^\mu \left\{\mathrm{tr}_4 \left[X^+ \left(\ln X^+ - 1\right)\right] + \mathrm{tr}_4 \left[X^- \left(\ln X^- - 1\right)\right]\right\} \quad \text{quantum spin} \tag{40}$$

Together with other kinetic-theory expressions, one obtains **tensor forms of thermodynamic relations** valid for any value of the spin polarization tensor ω

$$S^\mu_{\rm eq} = T^{\mu\alpha}_{\rm eq} \beta_\alpha - \frac{1}{2}\omega_{\alpha\beta} S^{\mu,\alpha\beta}_{\rm eq} - \xi N^\mu_{\rm eq} + \mathcal{N}^\mu, \qquad \mathcal{N}^\mu = \coth\xi\; N^\mu_{\rm eq} \neq P u^\mu \tag{41}$$

$$dS^\mu_{\rm eq} = -\xi dN^\mu_{\rm eq} + \beta_\lambda dT^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta} dS^{\mu,\alpha\beta}_{\rm eq} \quad \text{first law of thermodynamics} \tag{42}$$

$$d\mathcal{N}^\mu = N^\mu_{\rm eq} d\xi - T^{\lambda\mu}_{\rm eq} d\beta_\lambda + \tfrac{1}{2} S^{\mu,\alpha\beta}_{\rm eq} d\omega_{\alpha\beta} \quad \text{Gibbs-Duhem relations} \tag{43}$$

entropy conservation as a consequence of other conservation laws, very close similarity to MHD

PART 5: GOING OFF EQUILIBRIUM

W. Israel, J.M.Stewart, Annals Phys. 118 (1979) 341 & Phys.Lett. A58 (1976) 213
here we use the IS method to construct the Navier-Stokes theory

replacement of the equilibrium currents by the general ones (equilibrium + non-equilibrium corrections)

$$S^\mu = T^{\mu\alpha}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta} - \xi N^\mu + \mathcal{N}^\mu_{\rm eq} \tag{44}$$

Conservations laws, now for total angular momentum $J = L + S$

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu S^{\mu,\alpha\beta} = T^{\beta\alpha} - T^{\alpha\beta} \tag{45}$$

entropy production

$$\partial_\mu S^\mu \quad = \quad -\delta N^\mu \partial_\mu \xi + \delta T^{\mu\lambda}_s \partial_\mu \beta_\lambda + \delta T^{\mu\lambda}_a \left(\partial_\mu \beta_\lambda - \omega_{\lambda\mu}\right) - \frac{1}{2}\delta S^{\mu,\alpha\beta}\partial_\mu \omega_{\alpha\beta} \geq 0 \tag{46}$$

the second law of thermodynamics imposes constraints on non-equilibrium currents, they should be proportional to appropriate "gradients" multiplied by the kinetic coefficients

Generalized Tolman-Klein conditions define global equilibrium state

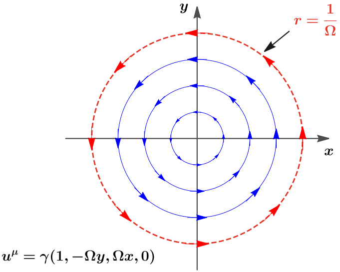
R. C. Tolman, Relativity, *Thermodynamics and Cosmology* (Oxford University Press, London, 1934)
O. Klein, Rev. Mod. Phys. 21 (1949) 531

$$\partial_\mu \xi = 0, \qquad \partial_{(\mu}\beta_{\lambda)} = 0, \qquad \omega_{\lambda\mu} = \partial_{[\mu}\beta_{\lambda]} = -\frac{1}{2}\left(\partial_\lambda\beta_\mu - \partial_\mu\beta_\lambda\right) \tag{47}$$

The middle equation, $\partial_\lambda\beta_\mu + \partial_\mu\beta_\lambda = 0$, is **the Killing equation** with a solution of the form

$$\beta^\mu = \beta_0^\mu + \varpi^{\mu\nu} x_\nu, \qquad \varpi^{\mu\nu} = -\varpi^{\nu\mu} = \mathrm{const}, \qquad \beta_0^\mu = \mathrm{const} \tag{48}$$

One possible solution: rigid rotation



L.D. Landau, E.M. Lifshitz , *Statistical Physics, Part 1* (Oxford Butterworth-Heinemann, 1980)

§ 26. **Rotating bodies**

In a state of thermal equilibrium, as we have seen in § 10, only a uniform translational motion and a uniform rotation of a body as a whole are possible. The uniform translational motion needs no special treatment, since by Galileo's relativity principle it has no effect on the mechanical properties of the body, nor therefore on its thermodynamic properties, and the thermodynamic quantities are unchanged except that the energy of the body is increased by its kinetic energy.

general definition of the statistical operator in QFT

$$e^{-(E-\mu)/T} \longrightarrow e^{-p\cdot\beta(x)+\xi(x)} \longrightarrow \widehat{\rho}_{\rm LEQ} = \frac{1}{Z}\exp\left[-\int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \frac{1}{2}\omega_{\lambda\nu}\widehat{J}^{\mu,\lambda\nu} - \xi\hat{N}^\mu\right)\right] \tag{49}$$

Σ is a space-like hypersurface, for example, corresponding to a constant LAB time t, in this case $\widehat{\rho}_{\rm LEQ} = \widehat{\rho}_{\rm LEQ}(t)$
in global equilibrium $\widehat{\rho}_{\rm LEQ}$ becomes independent of time

$$\varpi_{\lambda\mu} = \partial_{[\mu}\beta_{\lambda]} = -\frac{1}{2}\left(\partial_\lambda\beta_\mu - \partial_\mu\beta_\lambda\right) \quad \text{thermal vorticity is constant} \tag{50}$$

in local equilibrium $\widehat{\rho}_{\rm LEQ}$ depends on spacetime variables through

- thermal vorticity depending on space and time

$$\varpi_{\lambda\mu}(x) = \partial_{[\mu}\beta_{\lambda]}(x) = -\frac{1}{2}\left(\partial_\lambda\beta_\mu(x) - \partial_\mu\beta_\lambda(x)\right) \tag{51}$$

- thermal shear also depending on space and time

$$\xi_{\lambda\mu}(x) = \partial_{(\mu}\beta_{\lambda)}(x) = \frac{1}{2}\left(\partial_\lambda\beta_\mu(x) + \partial_\mu\beta_\lambda(x)\right) \tag{52}$$

- definition of $\Sigma_\mu(x)$

$\widehat{\rho}_{\rm LEQ}$ is used to calculate spin observables
succesful description of the data
F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, and A. Palermo, PRL127 (2021) 27, 272302

PART 6: SPIN DYNAMICS WITH REALISTIC HYDRODYNAMIC BACKGROUND

Is there a place for perfect spin hydrodynamics in RHIC?

S. K. Singh, R. Ryblewski, WF, Phys. Rev. C111 (2025) 024907

1. realistic 3D simulation of RHIC performed first, very good description of the rapidity distributions, transverse-momentum spectra, elliptic flow

   early stages, non-equilibrium processes, dissipation, **transfer between *L*** and ***S***

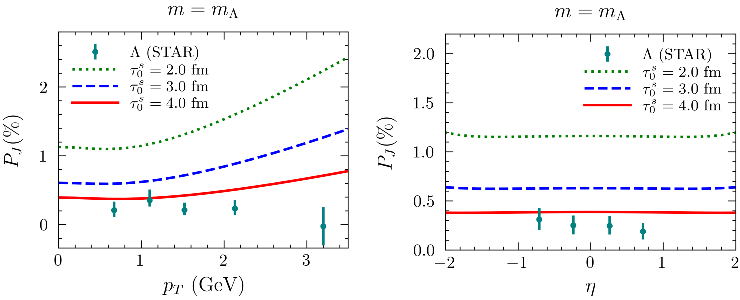
   late stages, spin approximately conserved, ***S*** ≈const

2. initialisation of the perfect spin hydrodynamics at the delayed proper time τ_0^S
   formula motivated by various independent works:
   S. Y. F. Liu and Y. Yin, JHEP 07 188
   F. Becattini, M. Buzzegoli, and A. Palermo, Phys. Lett. B820 (2021) 136519
   M. Buzzegoli, Phys. Rev. C105 (2022) 044907

$$\omega_{\mu\nu}(\tau_0^S) = \varpi^{\rm iso}_{\mu\nu} + 4\hat{\tau}_{[\mu}\xi^{\rm iso}_{\nu]\rho}u^\rho \qquad (53)$$

   $\varpi^{\rm iso}_{\mu\nu} = \frac{1}{T}\partial_{[\nu}u_{\mu]}$ is the isothermal part of thermal vorticity
   $\xi^{\rm iso}_{\mu\nu} = \frac{1}{T}\partial_{(\nu}u_{\mu)}$ represents the isothermal part of the thermal shear tensor $\xi_{\mu\nu} = \partial_{(\nu}\beta_{\mu)}$
   $\hat{\tau}^\mu = (1,0,0,0)$ is a fixed timelike vector which in Milne coordinates is normal to the constant-τ hypersurface

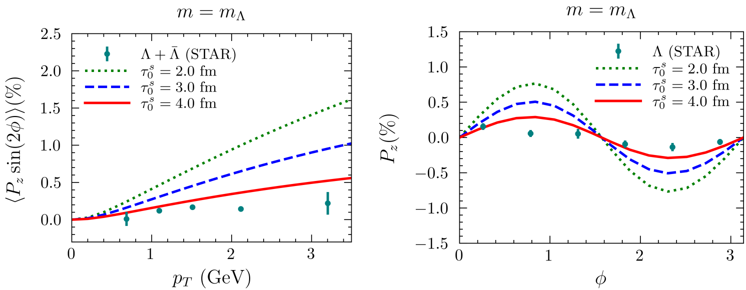
3. comparison with the data



Our numerical results for the component of Λ polarization along the orbital angular momentum direction for different initial time of spin evolution τ_0^s. Experimental data: STAR exp. at BNL, Au+Au collisions at $\sqrt{s_{\rm NN}}$= 200 GeV, c=20–60%

J. Adam et al. (STAR), Phys. Rev. C 98 (2018) 014910

J. Adam et al. (STAR), Phys. Rev. Lett. 123 (2019) 132301



Our numerical results for longitudinal Λ polarization for different initial time of spin evolution τ_0^s. Experimental data: STAR experiment at BNL, Au+Au collisions at $\sqrt{s_{\rm NN}}$= 200 GeV, c=20–60%

J. Adam et al. (STAR), Phys. Rev. C 98 (2018) 014910

J. Adam et al. (STAR), Phys. Rev. Lett. 123 (2019) 132301

Conclusions

1. The new formula for the equilibrium Wigner function eliminates, after more than 12 years, the deficiency of the most widely used expression so far. It eventually leads to a well-defined expression for the polarization magnitude.
2. Our expression is an extension of the basic formula given in the Landau-Lifshitz course.
3. If used to construct macroscopic currents, our equilibrium Wigner function allows us to obtain consistent thermodynamic relations derived in earlier studies.
4. For small polarization, our approach is consistent with the classical spin treatment based on the seminal work of Matthison.
5. We are able to verify nonlinear causality and symmetric hyperbolicity of the equations of motion of spin hydrodynamics constructed with our equilibrium function, which ensures local well-posedness of the initial value problem and stability of the theory.
6. The significance of the applicability criterion of the proposed framework is examined. The arguments are given that it does not constrain the real dynamics in heavy-ion collisions.

see the next talks by Valeriya Mykhaylova and Zbigniew Drogosz